III Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

In the previous chapter we analysed a first example of how to generalise the idea of integration. The amazing outcome was that this new integration process we defined to compute the length of a curve could be converted into a standard integral known to us to compute the area underneath graphs. We will now look at integration in more detail but before we do this we want to summarise the concept behind the standard integral process which defines the area underneath graphs since we we will later convert all the integrals we are going to define in higher dimensions into just standard integrals.

Let us consider the graph of a positive function $f(x)$ defined on an interval $[a, b]$ as in Fig. xiii. We partition the area underneath the graph of $f(x)$ and obtain a sum of the form $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ as an approximation for the area underneath the graph. ξ_i could either be x_{i-1} or x_i or indeed any arbitrary point in between x_{i-1} and x_i. In any case, taking the limit for the differences $x_i - x_{i-1}$ tending to 0 we obtain the area underneath the graph. We denote this limit by $\lim_{\|\mathcal{Z}\|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \int_a^b f(x)dx$. Mathematicians call this integral the *Riemann Integral*[a,b].

A sum of the type $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ is called a *Riemann sum* and a limit of the type $\lim_{\|\mathcal{Z}\|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ is called a *Riemann limit*. For a Riemann sum ξ_i has to be somewhere in the interval $[x_{i-1}, x_i]$ but the exact position does not matter in the limit.

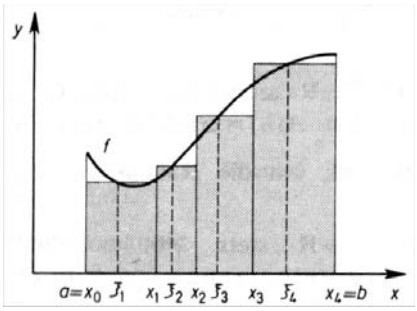


Fig. xiii The Riemann integral.

In this chapter our task will be to define integration mechanisms in higher dimensions but we will find that we will always be able to break down integration sums in higher dimensions into Riemann sums and therefore into Riemann integrals. Therefore it will be important to be able to recognise Riemann sums:

13 Riemann sum

A Riemann sum is a sum of the type

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \tag{40}$$

where all $\xi_i \in [x_{i-1}, x_i]$.

We will now investigate *Riemann-type* sums defined on curves, surfaces and volumes in $\mathbb{R}^2$ or $\mathbb{R}^3$ over real valued and also vector valued functions. This will allow us to define integrals along curves, surfaces and over volumes and relate these new integrals to ordinary one dimensional Riemann integrals.

[a] Georg Friedrich *Bernhard* Riemann, 1826-1866.

[b] There is another important concept of integration, the *Lebesgue Integral*, named after *Henri* Léon Lebesgue (1875-1941). This will be studied in future Analysis courses. For our applications the Riemann integral is sufficient.

3 Line integrals

We will now look at a different type of integral over vector valued functions which we call *line integrals*. The energy E a point mass gains by moving along a homogeneous force F is $F\Delta x$ where Δx denotes the distance the point mass is moving in direction of the force. This assumes that the point mass is moving parallel to the force, but what if we force it to move at an angle to the force? In this case it is only the component of the force parallel to the movement of the point mass that results in a gain of energy: $F\Delta x \cos\alpha$ where α is the angle between the direction of the force and the movement of the point mass. We can write this more conveniently as the scalar product between the vector $\mathbf{F}$ describing the force and the vector $\mathbf{\Delta x}$, the vector describing the movement: $E = \mathbf{F}.\mathbf{\Delta x} = F\Delta x \cos\alpha$. But what do we do if the force is not homogeneous? In this case the point mass might experience a different force (in terms of strength as well as direction) at every point along the path describing the movement. It now seems reasonable to solve this problem again using a Riemann-type limit where we look at sequences of partitions of the path describing the movement. We then again assume that between two partition points the movement is along a straight line and in addition along this straight line the force is homogeneous, taking the value of the force at some point between these partition points for the whole of this little straight line. It seems natural that in the limit this will give the energy gained or required by moving the point mass through this force. Let us call such a (possibly inhomogeneous) force a *force field* or more generally we define *vector fields* to be vector valued functions assigning a vector from $\mathbb{R}^n$ to each point in $\mathbb{R}^n$ (or at least to a subset of $\mathbb{R}^n$).

14 **Vector fields**[c]
A vector valued function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called a vector field.

FIG. xiv shows the inhomogeneous two dimensional vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \sin x \\ \cos y \end{pmatrix} \qquad (41)$$

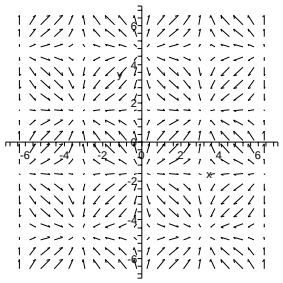
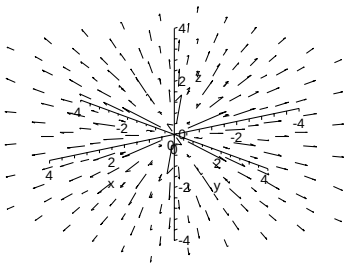


FIG. xiv Two dimensional vector field.

[c]Note that for a vector field the domain and the range have to have the same dimension.



The three dimensional vector field

$$\mathbf{F}(\mathbf{x}) \quad = \quad \begin{pmatrix} \frac{x}{r^3} \\ \frac{y}{r^3} \\ \frac{z}{r^3} \end{pmatrix} \qquad (42)$$

shown in FIG. xv has a spherical symmetry.

FIG. xv Three dimensional vector field.

Motivated by the idea of summing up scalar products in order to calculate the energy gained by the movement of a point mass through a force field we partition the path $\boldsymbol{\gamma}(t)$ and obtain the sum

$$\sum \mathbf{F}(\boldsymbol{\gamma}(t_i)).(\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1})) \ . \qquad (43)$$

As before, we now study the limit of the sum in EQ. 43 for the sequence of partitions $\mathcal{Z}$ of $[a, b]$ with $\|\mathcal{Z}\| \to 0$. We call this limit the *line integral* of $\mathbf{F}$ along the path $\boldsymbol{\gamma}$.

15 Line integrals

Given a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$*, and a path* $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$*. The line integral of* $\mathbf{F}$ *along the path* $\boldsymbol{\gamma}$ *is defined*

$$\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{x}).d\mathbf{x} \quad := \quad \lim_{\|\mathcal{Z}\|\to 0} \sum \mathbf{F}(\boldsymbol{\gamma}(t_i)).(\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1})) \ . \qquad (44)$$

An alternative notation for the line integral is

$$\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{x}).d\mathbf{x} \quad = \quad \int_{\boldsymbol{\gamma}} F_1(\mathbf{x})dx_1 + F_2(\mathbf{x})dx_2 + \ldots + F_n(\mathbf{x})dx_n \ . \qquad (45)$$

Using Taylor's theorem to first order on the increments $\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1})$ the Riemann-type sum $\sum \mathbf{F}(\boldsymbol{\gamma}(t_i)).(\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1}))$ can be written to first order as

$$\sum_i F(\boldsymbol{\gamma}(t_i))\boldsymbol{\gamma}'(t_i)(t_i - t_{i-1}) \ , \qquad (46)$$

which we now recognise as a Riemann sum.

16 Line integral over continuous vector fields along differentiable paths

$\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ *is a differentiable path then the line integral can be evaluated using a simple Riemann integral:*

$$\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{x}).d\mathbf{x} \quad = \quad \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)).\boldsymbol{\gamma}'(t)dt \ . \qquad (47)$$

Using the alternative notation we can write EQ. 47 *in the following form*[d]*:*

$$\int_{\boldsymbol{\gamma}} F_1(\mathbf{x})dx_1 + \ldots + F_n(\mathbf{x})dx_n \quad = \quad \int_a^b F_1(\boldsymbol{\gamma}(t))\frac{d\gamma_1}{dt}dt + \ldots + \int_a^b F_n(\boldsymbol{\gamma}(t))\frac{d\gamma_n}{dt}dt \ .$$

[d]Sometimes $\int_a^b F_1(\boldsymbol{\gamma}(t))\frac{d\gamma_1}{dt}dt + \ldots$ is shortened to $\int_a^b F_1 dx_1 + \ldots$. Note the danger of this notation: $\int F_i dx_i$ still has to be taken along the path and does not mean that $x_1, \ldots, x_n$ can be considered as independent variables for the integration.

Example 3.1 *Let us consider the vector field* $\mathbf{F}(\mathbf{x}) := \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)^T$ *given in* FIG. XV *and the continuous path* $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)^T$ *along the circular helix shown in* FIG. X *for* $t \in [0, 2\pi]$ *we find*

$$\int_\gamma \mathbf{F}.d\mathbf{x} = \int_0^{2\pi} \begin{pmatrix} \frac{\cos t}{(1+t^2)^{\frac{3}{2}}} \\ \frac{\sin t}{(1+t^2)^{\frac{3}{2}}} \\ \frac{t}{(1+t^2)^{\frac{3}{2}}} \end{pmatrix} . \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} dt = \int_0^{2\pi} \frac{t}{(1+t^2)^{\frac{3}{2}}} dt = 1 - \frac{1}{1+4\pi^2} \, . \tag{48}$$

It is obvious that different Jordan descriptions of the same Jordan arc lead to the same line integral since the reparametrisations simply correspond to a substitution of a strictly monotonous function. When calculating a line integral it is therefore essential to try to choose a suitable parametrisation such that the resulting Riemann integral becomes simple. The following integration rules are obvious and are given without proof.

17 Integration rules for line intergals

$$\int_\gamma (\alpha\mathbf{F} + \beta\mathbf{G}).d\mathbf{x} = \alpha \int_\gamma \mathbf{F}.d\mathbf{x} + \beta \int_\gamma \mathbf{G}.d\mathbf{x} \, , \tag{49}$$

$$\int_{\gamma_1 \oplus \gamma_2} \mathbf{F}.d\mathbf{x} = \int_{\gamma_1} \mathbf{F}.d\mathbf{x} + \int_{\gamma_2} \mathbf{F}.d\mathbf{x} \, , \tag{50}$$

$$\int_\gamma \mathbf{F}.d\mathbf{x} = -\int_{\gamma^-} \mathbf{F}.d\mathbf{x} \quad \text{with} \quad \boldsymbol{\gamma}^-(t) = \boldsymbol{\gamma}(b-t) \, , \tag{51}$$

$$\left| \int_\gamma \mathbf{F}.d\mathbf{x} \right| \leq \max_{t\in[a,b]} \|\mathbf{F}(\boldsymbol{\gamma}(t))\| \int_\gamma ds \, . \tag{52}$$

Example 3.2 *Let* $\mathbf{F}$ *be the force acting on a particle moving along the curve* $\mathcal{C}$*. The work done by the force in moving the particle an infinitesimal distance* $d\mathbf{x}$ *is* $\mathbf{F}.\,d\mathbf{x}$*; this is the generalisation to 3D of the 1D formula*

$$\textit{work done} = \textit{force} \times \textit{distance}.$$

Hence the total work done by the force in transversing the curve $\mathcal{C}$ *is*

$$W = \int_{\mathcal{C}} \mathbf{F}.\, d\mathbf{x}\,.$$

We can parametrise the curve by the time t *at which the particle is at a point* $\mathbf{x}$*:*

$$W = \int_{t_a}^{t_b} \mathbf{F}.\, \frac{d\mathbf{x}}{dt}\, dt = \int_{t_a}^{t_b} \mathbf{F}.\, \mathbf{v} dt$$

which is the time integral of the rate of doing work.

Using Newton's second law of motion $\mathbf{F} = m\ddot{\mathbf{x}}$ *(it is easiest to think of the case when there*

is just one force, e.g. gravity or rocket propulsion in deep space), gives

$$\begin{aligned} W &= \int_{t_a}^{t_b} \left(m\frac{d^2\mathbf{x}}{dt^2} \right) . \frac{d\mathbf{x}}{dt}\, dt \\ &= \int_{t_a}^{t_b} \frac{1}{2} m \frac{d}{dt} \left(\frac{d\mathbf{x}}{dt} . \frac{d\mathbf{x}}{dt} \right) dt \\ &= \left[\frac{1}{2} m \left(\frac{d\mathbf{x}}{dt} . \frac{d\mathbf{x}}{dt} \right) \right]_{t_a}^{t_b} \\ &= \textit{gain in kinetic energy.} \end{aligned}$$

Example 3.3 *Let*

$$\mathbf{F}(\mathbf{x}) = (y, x, 0)\,.$$

We will evaluate the line integral $\int \mathbf{F}.d\mathbf{x}$ *for the curve* $\mathcal{C}$ *parametrised by* $\mathbf{x} = (t, t^a, z(t))$ *where* $a > 0$ *and* $0 \le t \le 1$. *(Obviously, it does not matter what* $z(t)$ *is). We have*

$$\int_{\mathcal{C}} \mathbf{F}.\, d\mathbf{x} = \int_0^1 \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt = \int_0^1 (t^a + t\, a\, t^{a-1})\, dt = \int_0^1 (a+1)t^a\, dt = 1\,.$$

Note that the answer is independent of a, *i.e. the same result is obtained for different paths. In general, if different paths join the same end points, then the corresponding line integrals are not equal; as we shall explain in the following chapter they are in this case because* $ydx + xdy$ *is an exact differential(i.e. because* $\mathbf{F}(\mathbf{x})$ *can be written as the gradient of a function* $\phi(\mathbf{x})$*).*

We can verify this result for a straight line path from $(0,0,0)$ *to* $(1,1,0)$ *along paths parallel to the axes. Let*

$$\begin{aligned} \mathcal{C}_1 &: \quad x = t_1,\ y = 0 \quad \textit{for} \quad 0 \le t_1 \le 1\,, \\ \mathcal{C}_2 &: \quad x = 1,\ y = t_2 \quad \textit{for} \quad 0 \le t_2 \le 1\,. \end{aligned}$$

On $\mathcal{C}_1$

$$\mathbf{F}.\, d\mathbf{x} = \mathbf{F}.\, \frac{d\mathbf{x}}{dt_1} dt_1 = (0, t_1, 0).(1, 0, 0)\, dt_1 = 0\,.$$

On $\mathcal{C}_2$

$$\mathbf{F}.\, d\mathbf{x} = \mathbf{F}.\, \frac{d\mathbf{x}}{dt_2} dt_2 = (t_2, 1, 0).(0, 1, 0).\, dt_2 = dt_2\,.$$

Hence

$$\int_{\mathcal{C}_1 + \mathcal{C}_2} \mathbf{F}.\, d\mathbf{x} = \int_0^1 dt_2 = 1\,.$$

4 Plane surface integrals in $\mathbb{R}^2$ and volume integrals in $\mathbb{R}^3$

We will now proceed to defining integration over volumes and start with looking at functions defined on plane rectangles $\mathcal{R} := [a, b] \times [c, d]$ in $\mathbb{R}^2$. Let us therefore consider a real valued function $f(x, y) : \mathcal{R} \to \mathbb{R}$ on this rectangle $\mathcal{R}$.

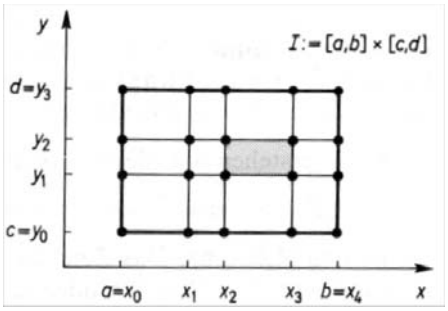


FIG. xvi Partitioning the rectangle $\mathcal{R}$.

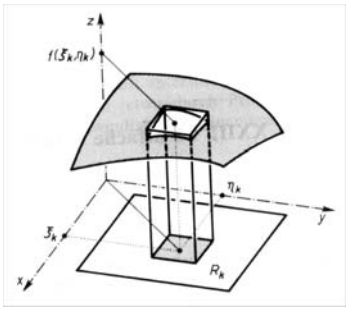


FIG. xvii Graph of $f(x,y)$ over the rectangle $\mathcal{R}$.

Following similar ideas as before we will partition the rectangle $\mathcal{R}$ into small rectangles as shown in FIG. xvi. We therefore take a partition $\mathcal{Z}_x$ of the x-axis and a partition $\mathcal{Z}_y$ of the y-axis with partition lengths $\|\mathcal{Z}_x\|$ and $\|\mathcal{Z}_y\|$ respectively. In case $f(x,y) \geq 0$ on the rectangle $\mathcal{R}$ then it is obvious from FIG. xvii that for small $\|\mathcal{Z}_x\|$ and $\|\mathcal{Z}_y\|$ the double sum

$$\sum_{i,j} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) \tag{53}$$

is an approximation for the volume underneath the graph of $f(x,y)$. ξ_i and η_j are as before chosen arbitrarily in the intervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ respectively. It is now very natural to consider the limit of this sum when the partition lengths $\|\mathcal{Z}_x\|$ and $\|\mathcal{Z}_y\|$ both tend to 0 independently.

18 Integrals over a rectangle

The real valued function $f(x,y)$ is defined on the rectangle $\mathcal{R} = [a,b] \times [c,d]$. The integral $\int_{\mathcal{R}} f(x,y)d(x,y)$ is the limit

$$\int_{\mathcal{R}} f(x,y)d(x,y) := \lim_{\|\mathcal{Z}_x\|,\|\mathcal{Z}_y\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) \ , \tag{54}$$

with $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$. Instead of $\int_{\mathcal{R}} f(x,y)d(x,y)$ we may also use the notation $\int_{\mathcal{R}} f(x,y)dS$.

We will now turn to the question how to calculate these new limits $\lim_{\|\mathcal{Z}_x\|,\|\mathcal{Z}_y\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1})$ and find that they can be written as two successive Riemann integrals:

$$\begin{aligned}
\int_{\mathcal{R}} f(x,y)d(x,y) &= \lim_{\|\mathcal{Z}_x\|,\|\mathcal{Z}_y\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) \\
&= \lim_{\|\mathcal{Z}_x\| \to 0} \left\{ \lim_{\|\mathcal{Z}_y\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)(y_j - y_{j-1}) \right\} (x_i - x_{i-1}) \qquad (55) \\
&= \lim_{\|\mathcal{Z}_x\| \to 0} \int_c^d f(\xi_i, y)dy \ \ (x_i - x_{i-1}) = \int_a^b \left(\int_c^d f(x,y)dy \right) dx \\
&=: \int_a^b \int_c^d f(x,y)dy\,dx \ .
\end{aligned}$$

This of course assumes that the inner integrals $\int_c^d f(x,y)dy$ exist for all $x \in [a,b]$. In the same way we could of course first integrate along the x-axis and then along the y-axis provided that

in this case the inner integrals $\int_a^b f(x,y)dx$ exist for all $y \in [c,d]$. The main achievement of our finding is that this again gives us a very simple way of calculating the complicated double limits $\int_{\mathcal{R}} f(x,y)d(x,y)$ in terms of two successive simple one dimensional Riemann integrals. This result is known as *Fubini's theorem*[e].

19 Fubini's theorem in two dimensions
The real valued function $f(x,y)$ is defined on the rectangle $\mathcal{R} = [a,b] \times [c,d]$. If the integrals $\int_c^d f(x,y)dy$ exist for all $x \in [a,b]$ then

$$\int_{\mathcal{R}} f(x,y)d(x,y) = \int_a^b \int_c^d f(x,y)dy\,dx\ , \tag{56}$$

and if the integrals $\int_a^b f(x,y)dx$ exist for all $y \in [c,d]$ then

$$\int_{\mathcal{R}} f(x,y)d(x,y) = \int_c^d \int_a^b f(x,y)dx\,dy\ . \tag{57}$$

Example 3.4 *Let us consider the real valued function $f(x,y) = xye^{-(x^2+y^2)}$ on the rectangle $\mathcal{R} = [1,4] \times [1,2]$. The graph of $f(x,y)$ is given in* FIG. xviii *The integral $\int_{\mathcal{R}} xye^{-(x^2+y^2)}d(x,y)$ computes the volume underneath the graph. We obtain*

$$\begin{aligned}\int_{\mathcal{R}} xye^{-(x^2+y^2)}d(x,y) &= \int_1^2 \int_1^4 xye^{-(x^2+y^2)}dx\,dy = \int_1^2 \left[-\frac{ye^{-(x^2+y^2)}}{2}\right]_{x=1}^{x=4} dy \\ &= \int_1^2 \frac{ye^{-(y^2+1)} - ye^{-(y^2+16)}}{2}dy = \frac{e^{-20} - e^{-17} - e^{-5} + e^{-2}}{4}\ .\end{aligned}$$

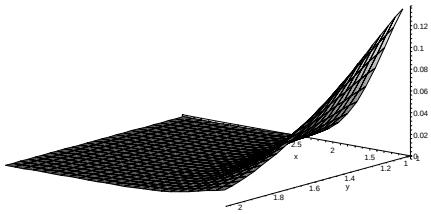


FIG. xviii Graph of $f(x,y) = xye^{-(x^2+y^2)}$.

We can now easily generalise integrals over rectangles to integrals over cuboids[f] in $\mathbb{R}^3$.

20 Integrals over 3-dimensional cuboids
The real valued function $f(x,y,z)$ is defined on the cuboid $\mathcal{Q} = [a,b] \times [c,d] \times [e,f]$. The integral $\int_{\mathcal{Q}} f(x,y,z)d(x,y,z)$ is the limit

$$\int_{\mathcal{Q}} f(x,y,z)d(x,y,z) := \lim_{\|\mathcal{Z}_x\|,\|\mathcal{Z}_y\|\mathcal{Z}_z\|\|\to 0} \sum_{i,j,k} f(\xi_i,\eta_j,\mu_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\ , \tag{58}$$

with $\xi_i \in [x_{i-1},x_i]$, $\eta_j \in [y_{j-1},y_j]$ and $\mu_k \in [z_{k-1},z_k]$. Instead of $\int_{\mathcal{Q}} f(x,y,z)d(x,y,z)$ we may also use the notation $\int_{\mathcal{Q}} f(x,y,z)dV$.

[e]Guido Fubini, 1879-1943.

[f]This could very easily be generalised to integrals over *n-dimensional intervals* $[a_1,b_1] \times [a_2,b_2] \times \ldots [a_n,b_n]$ in $\mathbb{R}^n$ but we want to restrict ourselves to $n = 2$ and $n = 3$.

How can we interpret such a 3-dimensional integral over a cuboid? We can imagine that the cuboid is made out of an inhomogeneous material with the mass (volume) density $f(\mathbf{x})$. Then $f(\xi_i, \eta_j, \mu_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$ is an approximation for the masses of the small cuboids given by the partitions $\mathcal{Z}_x$, $\mathcal{Z}_y$ and $\mathcal{Z}_z$ and therefore the sum $\sum_{i,j,k} f(\xi_i, \eta_j, \mu_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$ would obviously tend to the mass of the cuboid in the limit $\lim_{\|\mathcal{Z}_x\|,\|\mathcal{Z}_y\|\mathcal{Z}_z\|\|\to 0}$.

Fubini's theorem allows us again to break these integrals down into successive Riemann integrals:

21 **Fubini's theorem**
The real valued function $f(x, y, z)$ is defined on the cuboid $\mathcal{Q} = [a, b] \times [c, d] \times [e, f]$. If the integrals $\int_e^f f(x, y, z)dz$ and $\int_c^d \int_e^f f(x, y, z)dzdy$ exist for all $x \in [a, b]$ and all $y \in [c, d]$, then

$$\int_{\mathcal{Q}} f(x, y, z)d(x, y, z) = \int_a^b \int_c^d \int_e^f f(x, y, z)dz\, dy\, dx \; . \tag{59}$$

We could have obviously chosen a different order of integration in [21]. Fubini's theorem therefore tells us in particular that the order of integration does not matter.

22 **Exchange of order of integration**
The function $f(x, y)$ is integrable on the rectangle $\mathcal{R} = [a, b] \times [c, d]$ then

$$\int_a^b \int_c^d f(x, y)dy\, dx \;\; = \;\; \int_c^d \int_a^b f(x, y)dx\, dy = \int_{\mathcal{R}} f(x, y)d(x, y) \; , \tag{60}$$

provided all inner integrals $\int_a^b f(x, y)dx$ and $\int_c^d f(x, y)dy$ exist for all $y \in [c, d]$ and all $x \in [a, b]$ respectively.

Equation Eq. 60 can of course also be used for more than two successive integrals. For example a 3-dimensional integral $\int_{\mathcal{Q}} f(x, y, z)dV$ is independent of the order of integration provided all the successive integrals exist for all the required values of x, y, z. We call $\int_a^b \int_c^d f(x, y)dy\, dx$ and $\int_c^d \int_a^b f(x, y)dx\, dy$ the *iterated integrals* of $\int_{\mathcal{R}} f(x, y)d(x, y)$. The condition that all the successive integrals of the iterated integral have to exist should not be underestimated. It can for example very easily be shown that the two iterated integrals $\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dy\right] dx$ and $\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right] dy$ both exist but their values are in fact different. The reason why Fubini's theorem cannot be used in this case is simply that the inner integrals $\int_0^1 \frac{x-y}{(x+y)^3}\, dy$ and $\int_0^1 \frac{x-y}{(x+y)^3}\, dx$ do not exist for $x = 0$ or $y = 0$, respectively and therefore the conditions of [19] do not hold.

5 Integration over plane surfaces and Volumes

We can now integrate over rectangles in $\mathbb{R}^2$ and also over cuboids in $\mathbb{R}^3$, but how useful is that? What if we want to integrate over some different plane surface rather than a rectangle or some different volume rather than a cuboid? This can very easily be incorporated in the theory developed in Sec. 4 by using the characteristic function of a set $\mathcal{B} \subset \mathbb{R}^n$

$$\chi_{\mathcal{B}}(\mathbf{x}) := \begin{cases} 1 & , \mathbf{x} \in \mathcal{B} \\ 0 & , \mathbf{x} \notin \mathcal{B} \end{cases} \; . \tag{61}$$

Given a bounded[g] set $\mathcal{B} \subset \mathbb{R}^n$, we can then find a rectangle ($n = 2$) or a cuboid ($n = 3$) which completely contains $\mathcal{B}$. We then continue the function $f(\mathbf{x})$ defined on B to $\mathcal{R}$ by setting the function values outside B simply to 0. We define the integral of $f(\mathbf{x})$ over the set $\mathcal{B}$ as the integral of the function $f(\mathbf{x})\chi_{\mathcal{B}}(\mathbf{x})$ over[h] $\mathcal{R}$. In this way we can define an integral over general plane surfaces and general volumes in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively simply by using the theory developed in SEC. 4.

23 Integrals over plane surfaces

The integral of the real valued function $f(x, y)$ over the set $\mathcal{S}$ is[i]*:*

$$\int_{\mathcal{S}} f(x,y)dS \quad := \quad \int_{\mathcal{R}} f(x,y)\chi_{\mathcal{S}}(x,y)\, dS \,, \tag{62}$$

where $\mathcal{R}$ is a rectangle completely containing $\mathcal{S}$. In particular

$$\|\mathcal{S}\| := \int_{\mathcal{S}} dS \,, \tag{63}$$

is called the area of $\mathcal{S}$.

24 Integrals over volumes

The integral of the real valued function $f(x, y, z)$ over the set $\mathcal{V}$ is[j]*:*

$$\int_{\mathcal{V}} f(x,y,z)dV \quad := \quad \int_{\mathcal{R}} f(x,y,z)\chi_{\mathcal{V}}(x,y,z)\, dV \,, \tag{64}$$

where $\mathcal{Q}$ is a cuboid completely containing $\mathcal{V}$. In particular

$$\|\mathcal{V}\| := \int_{\mathcal{V}} dV \,, \tag{65}$$

is called the volume of $\mathcal{V}$.

It should be noted that for these integrals to exist we do not only require the functions $f(x, y)$ or $f(x, y, z)$ to be well-behaved but also the functions $\chi_{\mathcal{S}}(x, y)$ or $\chi_{\mathcal{V}}(x, y, z)$ need to be well-behaved. This basically means that the sets $\mathcal{S}$ or $\mathcal{V}$ need to have a smooth (or at least piecewise smooth) boundary. For this course we shall simply assume that the sets we want to integrate over satisfy this condition and leave the details to future Analysis courses. It is also immediately obvious how these integrals could be generalised to n-dimensional integrals. Mathematicians call n-dimensional sets which are integrable in this sense *Jordan measurable* and the corresponding n-dimensional volumes are called the *Jordan measure*.

It is clear why $\int_{\mathcal{S}} d(x, y)$ is defined to be the *area* of the surface $\mathcal{S}$ since the integral is the limit of sums $\sum_{i,j}(x_i - x_{i-1})(y_j - y_{j-1})$ over the surface $\mathcal{S}$. Some of these rectangles may go slightly outside the surface $\mathcal{S}$ but in the limit the sum over the rectangles described by $(x_i - x_{i-1})(y_j - y_{j-1})$ will approximate the area of the surface. In the same way we understand why $\int_{\mathcal{V}} d(x, y, z)$ describes a *volume*. It is also immediately obvious from FIG. xvii that the interpretation of $\int_{\mathcal{S}} f(\mathbf{x})d(x, y)$ as the volume underneath the graph of $f(\mathbf{x})$ coincides with the definition [24] of the volume $\int_{\mathcal{V}} d(x, y, z)$ as in FIG. xvii.

---

[g] The theory easily generalises to infinite sets but for our applications the bounded case is sufficient.

[h] Note that this definition does not really depend on $\mathcal{R}$ as long as $\mathcal{B} \subset \mathcal{R}$, and also that the function values outside $\mathcal{B}$ are not really needed.

[i] Instead of $\int_{\mathcal{S}} f(x, y)dS$ we may also frequently use the notations $\int_{\mathcal{S}} f(x, y)d(x, y)$.

[j] Instead of $\int_{\mathcal{V}} f(x, y, z)dV$ we may also frequently use the notation $\int_{\mathcal{V}} f(x, y, z)d(x, y, z)$.

25 Integration rules

$$\int_{\mathcal{V}} (\alpha f(\mathbf{x}) + \beta g(\mathbf{x}))dV = \alpha \int_{\mathcal{V}} f(\mathbf{x})dV + \beta \int_{\mathcal{V}} g(\mathbf{x})dV\,, \tag{66}$$

$$\int_{\mathcal{V}_1 \cup \mathcal{V}_2} f(\mathbf{x})dV = \int_{\mathcal{V}_1} f(\mathbf{x})dV + \int_{\mathcal{V}_2} f(\mathbf{x})dV\,, \tag{67}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint or at least disjoint up to boundary points. Similar rules apply for plane surface integrals.

We can now easily apply Fubini's theorem in order to evaluate these integrals:

Example 3.5 *Let us take in two dimensions the plane surface $\mathcal{S}$ bounded by the unit circle $x^2+y^2=1$ and by $x, y \geq 0$. How do we calculate the integral $\int_{\mathcal{S}} f(x,y)\, d(x,y)$ of the real valued function $f(x,y) = xy^2$ over $\mathcal{S}$?*

We embed $\mathcal{S}$ into the square $\mathcal{R} = [0,1] \times [0,1]$ and use Fubini's theorem for the integral of $f(\mathbf{x})\chi_{\mathcal{S}}(\mathbf{x})$ in order to first integrate parallel to the x-axis on the interval $[0,1]$ and then parallel to the y-axis also on $[0,1]$. However, since $f(\mathbf{x})\chi_{\mathcal{S}}(\mathbf{x})$ is 0 outside $\mathcal{S}$ it is clear that the upper limit of the integral parallel to the x-axis is not at 1 but actually at $\sqrt{1-y^2}$ and therefore depends on the actual value of y which could be anywhere in $[0,1]$.

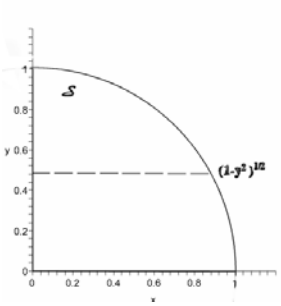


FIG. xix $\mathcal{S}$ bounded by $x^2+y^2=1$; $x, y \geq 0$.

We therefore obtain

$$\begin{aligned} \int_{\mathcal{S}} xy^2\, d(x,y) &= \int_0^1 \int_0^{\sqrt{1-y^2}} xy^2 dx\, dy \\ &= \int_0^1 \left[\frac{x^2y^2}{2}\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \int_0^1 \frac{y^2-y^4}{2} dy = \frac{1}{15}\,. \end{aligned} \tag{68}$$

It is now important to note that the upper and lower bounds of the inner integrals can depend on the variables of the outer integrals (but of course not vice versa). Instead of first integrating along x and then interacting along y we could have done it the other way round according to [22]. We would then obtain the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} xy^2 dy\, dx = \int_0^1 \frac{x(1-x^2)^{\frac{3}{2}}}{3} dx$ which will also give $\frac{1}{15}$ but is obviously more complicated.

Example 3.6 *In the same way we obtain in three dimensions three successive integrals with the boundaries of the inner integrals depending on the variables of the outer integrals. For example if the continuous function $f(x,y,z)$ is integrated over the volume $\mathcal{V}$ bounded by the ellipsoid $x^2+y^2+\frac{z^2}{4}=1$, then we can calculate $\int_{\mathcal{V}} f(x,y,z)d(x,y,z)$ in the following way: for fixed x and y the lower and upper bounds for the z co-ordinate are $-2\sqrt{1-x^2-y^2}$ and $2\sqrt{1-x^2-y^2}$. Projecting the ellipsoid into the xy-plane gives the circle $x^2+y^2=1$ and for fixed x we have y going from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$. Finally projecting the circle on the x-axis shows that x runs from -1 to 1. We therefore find $\int_{\mathcal{V}} f(x,y,z)d(x,y,z) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-y^2}}^{2\sqrt{1-x^2-y^2}} f(x,y,z)dz\, dy\, dx$.*

Most of the time the most difficult problem of integration over plane surfaces or volumes is actually describing the boundary of the plane surface of the volume correctly and easily in the successive integrals. As we have seen in our simple examples, by choosing a suitable order for the successive integrals the task of describing the lower and upper bounds could be simplified significantly which would also influence how complicated the successive integrations are. Therefore it is essential that we first think very carefully about a suitable order rather than picking a random order. Sometimes using Cartesian co-ordinates may not be the ideal solution for solving such an integral. We will study in SEC. 6 how a suitable substitution may simplify such an integral.

Finally, and just for completeness, we generalise integrals over real valued functions to integrals over vector valued functions simply by defining them as vectors of integrals over the real valued component functions.

26 **Integrals over vector valued functions**
The integral of the vector valued function $\mathbf{f}(\mathbf{x})$ *over the volume* $\mathcal{V}$ *is:*

$$\int_{\mathcal{V}} \mathbf{f}(\mathbf{x})\, dV \quad := \quad \begin{pmatrix} \int_{\mathcal{V}} f_1(\mathbf{x})\, dV \\ \vdots \\ \int_{\mathcal{B}} f_m(\mathbf{x})\, dV \end{pmatrix}, \tag{69}$$

and similarly for plane surface integrals.

6 Substitution rule

For the one dimensional Riemann integral the substitution rule

$$\int_{g(a)}^{g(b)} f(x)dx \quad = \quad \int_a^b f(g(t))g'(t)dt \tag{70}$$

plays a central rôle if we actually want to solve complicated integrals. In EQ. 70 the substitution function $g(t)$ is either strictly increasing or strictly decreasing which corresponds to $g'(t) > 0$ or $g'(t) < 0$ for all $t \in (a, b)$ respectively. Since our definition of integrals over higher dimensional functions follows the same ideas as in one dimension it is not surprising that a very similar substitution rule holds in higher dimensions:

27 **Substitution rule for integrals in** $\mathbb{R}^n$
If $\mathbf{g} : \mathcal{B} \to \hat{\mathcal{B}}$ *is a bijective transformation from the set* $\mathcal{B}$ *onto* $\hat{\mathcal{B}} = \mathbf{g}(\mathcal{B})$ *with either* $\det J_{\mathbf{g}}(\mathbf{t}) > 0$ *or* $\det J_{\mathbf{g}}(\mathbf{t}) < 0$ *for all* $\mathbf{t} \in \mathcal{B}$ *then*

$$\int_{\mathbf{g}(\mathcal{B})} f(\mathbf{x})\, d(x_1, \ldots, x_n) \quad = \quad \int_{\mathcal{B}} f(\mathbf{g}(\mathbf{t}))\ |\det J_{\mathbf{g}}(\mathbf{t})|\ d(t_1, \ldots, t_n)\ , \tag{71}$$

where $|\det J_{\mathbf{g}}(\mathbf{t})|$ *denotes the absolute value of the determinant of the Jacobi matrix of the transformation* $\mathbf{g}(\mathbf{t})$. $|\det J_{\mathbf{g}}(\mathbf{t})|$ *is called the Jacobian of the transformation* $\mathbf{g}(\mathbf{t})$.

We call the function $g(\mathbf{t})$ *a substitution function and for the Jacobian we use the notation*

$$\frac{d(x_1, \ldots, x_n)}{d(t_1, \ldots, t_n)} \quad := \quad |\det J_{\mathbf{g}}(\mathbf{t})|\ . \tag{72}$$

The fact that either $\det J_{\mathbf{g}}(\mathbf{t}) > 0$ or $\det J_{\mathbf{g}}(\mathbf{t}) < 0$ for all $\mathbf{t} \in \mathcal{B}$ makes sure that the transformation $\mathbf{g}(\mathbf{t})$ is invertible as explained in Part IA Differential Equations. This is completely analogous to $g'(t) > 0$ or $g'(t) < 0$ in Eq. 70. The reason why we have to use the absolute value $|\det J_{\mathbf{g}}(\mathbf{t})|$ rather than just $\det J_{\mathbf{g}}(\mathbf{t})$ is that by definition of the integral $\int_{\mathcal{B}} f(\mathbf{x}) d(x_1, \ldots, x_n)$ we cannot keep track of the fact that under the substitution certain co-ordinate axes may have inverted their directions and therefore we do not know if we need to introduce a suitable number of minus signs. For example, in the one dimensional Riemann case, if the substitution function $g(t)$ is strictly decreasing then the *lower limit* $g(a)$ is be larger than the *upper limit* $g(b)$ provided $a < b$. But the Riemann integral $\int_{g(a)}^{g(b)} f(x)dx$ corrects itself simply by using $\int_{g(a)}^{g(b)} f(x)dx = -\int_{g(b)}^{g(a)} f(x)dx$. The multidimensional integral $\int_{\mathbf{g}(\mathcal{B})} f(\mathbf{x}) d(x_1, \ldots, x_n)$ does not keep track of such changes of directions but by taking $|\det J_{\mathbf{g}}(\mathbf{t})|$ as Jacobian rather than $\det J_{\mathbf{g}}(\mathbf{t})$ just corrects a possible overall change in orientation.

Example 3.7 *The area of the disc $\mathcal{D}$: $x^2 + y^2 \leq R^2$ is given by $\int_{\mathcal{D}} d(x, y)$. Describing the disc in terms of Cartesian co-ordinates ultimately involves lower and upper limits of the form $\pm\sqrt{R^2 - x^2}$, it therefore seems much more natural to describe the disc in terms of plane polar co-ordinates: $r \in [0, R]$ and $\phi \in [0, 2\pi]$ describes $\mathcal{D}$. The Jacobi matrix of the transformation $x = r\cos\phi$ and $y = r\sin\phi$ is*

$$J = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} , \tag{73}$$

and therefore we find for the Jacobian $\frac{d(x,y)}{d(r,\phi)} = r$. Hence we find the well known formula for the area of a disc:

$$\|\mathcal{D}\| = \int_{\mathcal{D}} d(x, y) = \int_0^R \int_0^{2\pi} r \, d\phi \, dr = \pi R^2 . \tag{74}$$

28 **Transformation to plane polar coordinates**
The Jacobian for plane polar coordinates is

$$\frac{d(x, y)}{d(r, \phi)} = r . \tag{75}$$

The transformation to plane polar coordinates is then given by

$$\int_{\hat{\mathcal{S}}} f(x, y) \, d(x, y) = \int_{\mathcal{S}} f(r\cos\phi, r\sin\phi) \; r \; d(r, \phi) , \tag{76}$$

where $\hat{\mathcal{S}}$ and $\mathcal{S}$ describe the corresponding surface in the xy-plane or in the $r\phi$-plane respectively.

29 **Transformation to cylindrical polar coordinates**
The Jacobian for cylindrical polar coordinates is

$$\frac{d(x, y, z)}{d(\rho, \phi, z)} = \rho . \tag{77}$$

The transformation to cylindrical polar coordinates is then given by

$$\int_{\hat{\mathcal{V}}} f(x, y, z) \, d(x, y, z) = \int_{\mathcal{V}} f(\rho\cos\phi, \rho\sin\phi, z) \; \rho \; d(\rho, \phi, z) , \tag{78}$$

where $\hat{\mathcal{V}}$ and $\mathcal{V}$ describe the corresponding volume in xyz-space or in $\rho\phi z$-space respectively.

30 **Transformation to spherical polar coordinates**
The Jacobian for spherical polar coordinates is

$$\frac{d(x,y,z)}{d(r,\phi,\theta)} = r^2 \sin\theta \, . \tag{79}$$

The transformation to spherical polar coordinates is then given by

$$\int_{\hat{\mathcal{V}}} f(x,y,z)\, d(x,y,z) = \int_{\mathcal{V}} f(r\cos\phi\, \sin\theta, r\sin\phi\, \sin\theta, r\cos\theta)\; r^2 \sin\theta \; d(r,\phi,\theta) \, , \tag{80}$$

where $\hat{\mathcal{V}}$ and $\mathcal{V}$ describe the corresponding volume in xyz-space or in $r\phi\theta$-space respectively.

For many applications the coordinate transformation is defined through its inverse rather than through the function itself but actually inverting the coordinate transformation may prove very difficult. However, the transformation of the integral in Eq. 70 does not require us to explicitly invert the transformation, all we need is the determinant of the Jacobi matrix of the inverse transformation. In Part IA Differential Equations you found that the Jacobi matrix of the inverse of a function is just the inverse matrix of the Jacobi matrix of the function. We also know that the determinant of the inverse of a matrix is simply the reciprocal value of the determinant of the matrix. We therefore find:

31 **Jacobian of the inverse transformation**
If $\mathbf{g} : \hat{\mathcal{B}} \to \mathcal{B}$ is a substitution function then $g(\mathbf{t})$ is invertible and the inverse function $\mathbf{g}^{-1} : \mathcal{B} \to \hat{\mathcal{B}}$ is again a substitution function with Jacobian

$$\left|\det J_{\mathbf{g}^{-1}}(\mathbf{x})\right| = \frac{1}{\left|\det J_{\mathbf{g}}(\mathbf{t})\right|} \, , \tag{81}$$

or in other words:

$$\frac{d(x_1,\ldots,x_n)}{d(t_1,\ldots,t_n)} = \frac{1}{\frac{d(t_1,\ldots,t_n)}{d(x_1,\ldots,x_n)}} \, . \tag{82}$$

Examples

Example 3.8 *We will calculate, in two ways, the integral*

$$I = \int_{\mathcal{R}} y \, dS \, ,$$

where $\mathcal{R}$ is specified by

$$1 \le x \le 2 \quad \textit{and} \quad 1/x \le y \le e^x \, .$$

(i) Do the y-integration first. We have

$$\begin{aligned} I = \int_{x=1}^{2} \left(\int_{y=1/x}^{e^x} y \, dy \right) dx &= \int_{x=1}^{2} \left[\tfrac{1}{2} y^2\right]_{1/x}^{e^x} dx \\ &= \tfrac{1}{2} \int_1^2 (e^{2x} - 1/x^2) dx \\ &= \tfrac{1}{4}(e^4 - e^2 - 1) \end{aligned}$$

(ii) Do the x integral first. We have

$$\begin{aligned} I &= \int_{\frac{1}{2}}^{1} \left(\int_{1/y}^{2} y \, dx \right) dy + \int_{1}^{e} \left(\int_{1}^{2} y \, dx \right) dy + \int_{e}^{e^2} \left(\int_{\ln y}^{2} y \, dx \right) dy \\ &= \int_{\frac{1}{2}}^{1} y(2 - 1/y) dy + \int_{1}^{e} y \, dy + \int_{e}^{e^2} y(2 - \ln y) dy \\ &= \textit{same as the previous answer.} \end{aligned}$$

The huge brackets (parentheses) in the first line of each calculation can be omitted without creating ambiguity.

Example 3.9 *We will calculate, in two different ways, the integral*

$$I = \int_{\mathcal{V}} x^2 dV \ ,$$

where $\mathcal{V}$ is the tetrahedron bounded by the four planes

$$x = 0, \ y = 0, \ z = 0 \quad \textit{and} \quad x + y + z = a \, .$$

The vertices of the tetrahedron are at $(0, 0, 0)$, $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$.

(i) Do the x integral last. We have

$$\begin{aligned} I &= \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \, dz \, dy \, dx \\ &= \int_0^a \int_0^{a-x} x^2 (a - x - y) \, dy \, dx \\ &= \int_0^a \tfrac{1}{2} x^2 (a - x)^2 \, dx = \frac{a^5}{60}. \end{aligned}$$

(ii) Do the x integral first. We have

$$\begin{aligned} I &= \int_0^a \int_0^{a-z} \int_0^{a-z-y} x^2 \, dx \, dy \, dz \\ &= \int_0^a \int_0^{a-z} \tfrac{1}{3} (a - z - y)^3 dy \, dz \\ &= \int_0^a \tfrac{1}{12} (a - z)^4 dz = \frac{a^5}{60}. \end{aligned}$$

Example 3.10 *We will calculate the Gaussian integral I given by*

$$I = \int_0^\infty e^{-x^2} dx$$

by means of a remarkably cunning plan.

We consider

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-x^2 - y^2} dx dy$$

We now change coordinates to plane polars given by $x = \rho\cos\phi$, $y = \rho\sin\phi$. *First we calculate the Jacobian.*

$$\frac{\partial x}{\partial \rho} = \cos\phi\,, \qquad \frac{\partial x}{\partial \phi} = -\rho\sin\phi\,,$$
$$\frac{\partial y}{\partial \rho} = \sin\phi\,, \qquad \frac{\partial y}{\partial \phi} = \rho\cos\phi\,.$$

Hence

$$\frac{\partial(x,y)}{\partial(\rho,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \rho\cos^2\phi + \rho\sin^2\phi = \rho\,.$$

Thus

$$I^2 = \int_0^\infty \int_0^{\pi/2} e^{-\rho^2}\,\rho\,d\phi\,d\rho = \frac{\pi}{2}\int_0^\infty e^{-\rho^2}\,\rho\,d\rho = \frac{\pi}{4}$$

and $I = \frac{1}{2}\sqrt{\pi}$ *(taking the positive square root since the integrand is positive).*

Example 3.11 *We will evaluate*

$$I = \int_{\mathcal{R}} \frac{1}{x^2}\,dS\,,$$

where the region $\mathcal{R}$ *is bounded by the lines*

$$y = 0\,,\ y = x\,,\ x + y = 1 \quad \textit{and} \quad x + y = 2\,.$$

Since the boundaries can be expressed as either $y = kx$ *or* $x + y = c$, *this suggests trying the new variables*

$$u = x + y \quad \textit{and} \quad v = y/x\,.$$

The Jacobian is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2}\,,$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2}{x+y}\,.$$

Thus

$$I = \int_1^2 \int_0^1 \frac{1}{u}\,dv\,du = \ln 2\,.$$

Note that if instead we had chosen $u = y/x$ *and* $v = x + y$ *then the Jacobian would have changed sign since the handedness of the system would not have been preserved. We would no doubt have remembered to use the modulus of the Jacobian,*

Of course, we could have done the integral fairly easily without changing variables.

7 Surface integrals in $\mathbb{R}^3$

We studied line integrals in Sec. 3 motivated by the fact that the scalar product of force and a displacement vector gives us the energy gained or needed along the movement of a point mass. The scalar product plays a similar rôle if we study the flow of a fluid through a surface but this time we have a two dimensional problem. Let us assume that we observe a straight and

homogeneous flow of water with a flow density of f litres of water per second per metre square. If we then put an open tube straight into the flow, and the tube opening has a surface of s metres square, how much water will we collect through this tube? The answer is obviously fs litres of water per second.

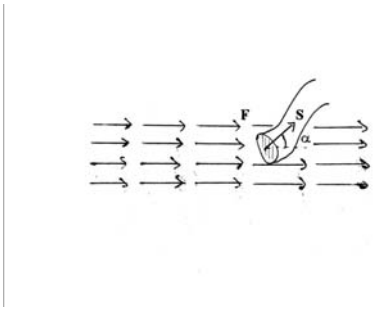


FIG. xx Flow through a surface $\mathcal{S}$.

But what if we put the tube into the flow at an angle α to the flow? Obviously the effective surface of the tube opening seen by the water flow is now reduced to $s\cos\alpha$ and therefore the flow through the tube is $fs\cos\alpha$. It therefore makes sense to assign a vector to the flow of water $\mathbf{F}$ with the direction of $\mathbf{F}$ pointing in the direction of the flow of water and the magnitude $\|\mathbf{F}\|$ being the flow density f. We also assign a vector to the surface of the opening of the tube $\mathbf{S}$ with a direction normal to the surface of the opening and the magnitude $\|\mathbf{S}\|$ being the area s.

By taking the scalar product $\mathbf{F}.\mathbf{S} = fs\cos\alpha$ we therefore obtain the flow of water through the opening of the tube even if it is held inside the flow at an angle α. But what if the flow is not homogeneous? We would therefore have a vector field $\mathbf{F}(\mathbf{x})$ describing the flow density of the water. $\mathbf{F}(\mathbf{x})$ might then vary with $\mathbf{x}$ in terms of direction as well as magnitude! If we then ask how much water will flow through the opening described by the vector $\mathbf{S}$ then by now it is obvious to us what we have to do: we partition the (two dimensional) surface into small surfaces $\mathcal{S}_{ij}$ described by the vectors $\mathbf{S}_{ij}$, evaluate $\mathbf{F}(\mathbf{x})$ somewhere on each $\mathcal{S}_{ij}$ at (ξ_i, η_j) and keep $\mathbf{F}(\xi_i, \eta_j)$ constant on the *surface element* $\mathbf{S}_{ij}$. Each of the vectors $\mathbf{S}_{ij}$ is normal to the corresponding $\mathcal{S}_{ij}$ and the magnitude matches the area of $\mathcal{S}_{ij}$. If the surface elements are *small* then the scalar product $\mathbf{F}(\xi_i, \eta_j).\mathbf{S}_{ij}$ obviously approximates the amount of water flowing through $\mathbf{S}_{ij}$ and summing over all these scalar products $\sum_{i,j} \mathbf{F}(\xi_i, \eta_j).\mathbf{S}_{ij}$ approximates the amount of water flowing through the whole opening. In case the shape of the opening and the vector field $\mathbf{f}(\mathbf{x})$ are both *well-behaved* then we hope that the limit of this sum under the partition lengths tending to 0 would exist and would therefore describe the flow of water through the opening of the tube exactly.

Let us now formalise this procedure. The opening of the tube in our example is of course a flat surface, which essentially means that all $\mathbf{S}_{ij}$ in the above example point in the same direction. But there is in fact no need for this surface to be flat and that in the sum $\sum_{i,j} \mathbf{F}(\xi_i, \eta_j).\mathbf{S}_{ij}$ all $\mathbf{S}_{ij}$ point in the same direction. Being able to generalise this procedure to surfaces which are not flat would allow us to calculate the *flow* of a vector field $\mathbf{F}(\mathbf{x})$ even through surfaces $\mathcal{S}$ in $\mathbb{R}^3$ which are not flat. The procedure we are going to follow is now straightforward generalisation of the integration techniques studies earlier but before we can start we will first need to define surfaces in a suitable way.

Let us assume that the vector valued function $\mathbf{\Phi}(u, v)$ describes the points on a surface $\mathcal{S}$ in $\mathbb{R}^3$. For example the function

$$\mathbf{\Phi}(\phi, \theta) \quad := \quad \begin{pmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{pmatrix} \tag{83}$$

describes the surface of a sphere with radius one, the *unit sphere*, in $\mathbb{R}^3$ as ϕ and θ go through the intervals $[0, 2\pi]$ and $[0, \pi]$ respectively. Another important example is that for a real valued

function $f(x,y) : D \to \mathbb{R}$ the vector valued function

$$\mathbf{\Phi}(x,y) \quad := \quad \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} \tag{84}$$

describes the surface in $\mathbb{R}^3$ which is given by the graph of the function $f(x,y)$ as (x,y) goes through the domain D.

The function $\mathbf{\Phi} : \mathcal{P} \subset \mathbb{R}^2 \to \mathbb{R}^3$ is defined on the set $\mathcal{P}$ which obviously has to be a subset of $\mathbb{R}^2$ in order to generate a two dimensional surface $\mathcal{S}$ in three dimensional space. We want to call $\mathcal{P}$ the *parameter space* of $\mathbf{\Phi}$ describing the surface $\mathcal{S}$. $\mathbf{\Phi}$ itself is called a *parametrisation* of the surface $\mathcal{S}$.

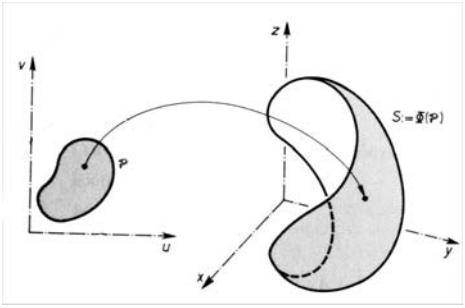


FIG. xxi Surface $\mathcal{S}$ with parameter space $\mathcal{P}$.

If we are now given a surface $\mathcal{S}$ with a parametrisation $\mathbf{\Phi}$ and parameter space $\mathcal{P}$ then we can easily partition $\mathcal{S}$ by partitioning $\mathcal{P}$ in the usual way. The partition of $\mathcal{P}$ will obviously result in a partition of $\mathcal{S}$. Let $\mathcal{Z}_u$ and $\mathcal{Z}_v$ denote the corresponding partitions on $\mathcal{P}$ with partition lengths $\|\mathcal{Z}_u\|$ and $\|\mathcal{Z}_v\|$. We now assume the surface to be flat on each surface element generated by this partition. As in our earlier example the vectors $\mathbf{S}_{ij}$ describe these small surface elements in the way that the direction of $\mathbf{S}_{ij}$ is normal to the surface element and the magnitude matches the area of the surface element. Assume now that the vector field $\boldsymbol{F}(\mathbf{x})$ is defined everywhere on the surface $\mathcal{S}$ then we are obviously interested in the following sums

$$\sum_{i,j} \mathbf{F}(\mathbf{\Phi}(\xi_i, \eta_j)).\mathbf{S}_{ij} \ , \tag{85}$$

where $\xi_i \in [u_{i-1}, u_i]$ and $\eta_j \in [v_{j-1}, v_j]$ and the u_is and v_js correspond to the partition of $\mathcal{P}$. We then consider the limit of EQ. 85 for the partition lengths $\|\mathcal{Z}_u\|$ and $\|\mathcal{Z}_v\|$ tending to 0 independently. We will call this integral the *surface integral* of $\mathbf{F}$ over the surface $\mathcal{S}$. We will see later that this limit is in fact independent of the chosen parametrisation $\mathbf{\Phi}$.

32 **Surface integrals over vector fields**
The function $\mathbf{\Phi} : \mathcal{P} \to \mathbb{R}^3$ is a parametrisation of the surface $\mathcal{S}$ with parameter space $\mathcal{P} \subset \mathbb{R}^2$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. The surface integral of $\mathbf{F}$ over the surface $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{F}(\mathbf{x}).d\mathbf{S} = \lim_{\|\mathcal{Z}_u\|,\|\mathcal{Z}_v\| \to 0} \sum_{i,j} \mathbf{F}(\mathbf{\Phi}(\xi_i, \eta_j)).\mathbf{S}_{ij} \ , \tag{86}$$

where ξ_i and η_j are in $[u_{i-1}, u_i]$ or $[v_{j-1}, v_j]$ respectively.

As before the definition [32] does not tell us how to actually compute the integral. Our hope is of course that this very complicated integral would once more translate into simple one

dimensional Riemann integrals (provided $\mathbf{F}$ and $\mathcal{S}$ are both well-behaved). Let us therefore look more closely at the surface elements $\mathbf{S}_{ij}$.

Let us assume that the surface element $\mathbf{S}_{ij}$ is flat. In addition, just like for the proof of the substitution rule [27], we approximate the surface element $\mathbf{S}_{ij}$ using the parallelogram spanned by the points $\mathbf{\Phi}(u_{i-1}, v_{j-1})$, $\mathbf{\Phi}(u_i, v_{j-1})$ and $\mathbf{\Phi}(u_{i-1}, v_j)$. In the limit $\|\mathcal{Z}_u\|$ and $\|\mathcal{Z}_v\|$ tending to 0 this should give the correct surface integral.

The parallelogram is spanned by the vectors $\mathbf{\Phi}(u_i, v_{j-1}) - \mathbf{\Phi}(u_{i-1}, v_{j-1})$ and $\mathbf{\Phi}(u_{i-1}, v_j) - \mathbf{\Phi}(u_{i-1}, v_{j-1})$. We know that the cross product of two vectors $\mathbf{a} \times \mathbf{b}$ gives a vector normal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ with the magnitude $\|\mathbf{a}\|\|\mathbf{b}\| \sin\alpha$ where $\sin\alpha$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. But $\|\mathbf{a}\|\|\mathbf{b}\| \sin\alpha$ is equal to the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Therefore we can take the surface element $\mathbf{S}_{ij}$ to be the cross product of $\mathbf{\Phi}(u_i, v_{j-1}) - \mathbf{\Phi}(u_{i-1}, v_{j-1})$ and $\mathbf{\Phi}(u_{i-1}, v_j) - \mathbf{\Phi}(u_{i-1}, v_{j-1})$:

$$\mathbf{S}_{ij} \quad := \quad (\mathbf{\Phi}(u_i, v_{j-1}) - \mathbf{\Phi}(u_{i-1}, v_{j-1})) \times (\mathbf{\Phi}(u_{i-1}, v_j) - \mathbf{\Phi}(u_{i-1}, v_{j-1})) \ . \tag{87}$$

We assume that $\mathbf{\Phi}$ is smooth such that we can use the Taylor expansion for the two increments of $\mathbf{\Phi}$ in Eq. 87:

$$\mathbf{\Phi}(u_i, v_{j-1}) - \mathbf{\Phi}(u_{i-1}, v_{j-1}) \quad = \quad \frac{\partial \mathbf{\Phi}(u_{i-1}, v_{j-1})}{\partial u}(u_i - u_{i-1}) + \dots \ , \tag{88}$$

$$\mathbf{\Phi}(u_{i-1}, v_j) - \mathbf{\Phi}(u_{i-1}, v_{j-1}) \quad = \quad \frac{\partial \mathbf{\Phi}(u_{i-1}, v_{j-1})}{\partial v}(v_i - v_{i-1}) + \dots \ . \tag{89}$$

We hence obtain:

$$\mathbf{S}_{ij} \quad = \quad \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} (u_i - u_{i-1})(v_j - v_{j-1}) \ . \tag{90}$$

It is now obvious that the surface integral can be converted into the plane two dimensional surface integral $\int_P \mathbf{F}(\mathbf{\Phi}(u,v)).\left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right) d(u,v)$ over the (flat) parameter space P.

33 Surface integrals over differentiable surfaces

The surface $\mathcal{S}$ is parametrised by the function $\mathbf{\Phi}$ on the parameter space P, then the surface integral of the vector field $\mathbf{F}$ over $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{F}(\mathbf{x}).d\mathbf{S} \quad = \quad \int_P \mathbf{F}(\mathbf{\Phi}(u,v)).\left(\frac{\partial \mathbf{\Phi}(u,v)}{\partial u} \times \frac{\partial \mathbf{\Phi}(u,v)}{\partial v}\right) d(u,v) \ . \tag{91}$$

For simplicity we use the notation

$$d\mathbf{S} \quad = \quad \frac{\partial \mathbf{\Phi}(u,v)}{\partial u} \times \frac{\partial \mathbf{\Phi}(u,v)}{\partial v} \, d(u,v) \ . \tag{92}$$

Just like for integrals along paths we of course need to assume that the parametrisation $\mathbf{\Phi}$ of the surface $\mathcal{S}$ is sufficiently smooth for the derivatives in Eq. 91 to exist. In case $\mathbf{\Phi}$ is only piecewise differentiable (i.e. it can be split up in finitely many differentiable surfaces) then we can simply integrate over each differentiable piece and add them all up. Therefore everything we are gong to say about differentiable parametrisations of surfaces is in this way also valid for piecewise differentiable parametrisations.

Example 3.12 *Let us for example consider the vector field $\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{r^3}$. We assume that $\mathbf{F}(\mathbf{x})$ describes the flow density of a fluid. How much of the fluid flows through the (open) upper*

hemisphere $x^2+y^2+z^2=R^2$, $z\geq 0$? *In order to find* $d\mathbf{S}$ *for the upper hemisphere we parametrise it using*

$$\mathbf{\Phi}(\mathbf{x}) \;=\; \begin{pmatrix} R\cos\phi\,\sin\theta \\ R\sin\phi\,\sin\theta \\ R\cos\theta \end{pmatrix}. \tag{93}$$

Taking the cross product of the two tangent vectors leads to

$$d\mathbf{S} \;=\; \frac{\partial\mathbf{\Phi}}{\partial\theta}\times\frac{\partial\mathbf{\Phi}}{\partial\phi} = \begin{pmatrix} R\cos\phi\,\cos\theta \\ R\sin\phi\,\cos\theta \\ -R\sin\theta \end{pmatrix}\times\begin{pmatrix} -R\sin\phi\,\sin\theta \\ R\cos\phi\,\sin\theta \\ 0 \end{pmatrix} = R^2\sin\theta\mathbf{e}_r d\theta\,d\phi\,. \tag{94}$$

We therefore obtain for the flow of the fluid through the upper hemisphere

$$\int_{\mathcal{S}}\mathbf{F}.d\mathbf{S} \;=\; \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{R\mathbf{e}_r}{R^3}R^2\sin\theta\,.\mathbf{e}_r\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\theta d\theta\,d\phi = 2\pi\,. \tag{95}$$

It remains to be shown that the definition [32] of the surface integral is indeed independent of the explicit choice of the parametrisation $\mathbf{\Phi}(u,v)$. If we assume that $\hat{\mathbf{\Phi}}(\hat{u},\hat{v})$ is a different parametrisation of the same surface $\mathcal{S}$ with parameter space $\hat{P}$, then we can obviously find a function $\boldsymbol{\kappa}: P\to\hat{P}$ such that $\mathbf{\Phi}=\hat{\mathbf{\Phi}}\circ\boldsymbol{\kappa}$. Using the chain rule and setting $\hat{u}=\kappa_1(u,v)$ and $\hat{v}=\kappa_2(u,v)$ we find

$$\frac{\partial\mathbf{\Phi}}{\partial u}=\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{u}}\frac{\partial\kappa_1}{\partial u}+\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{v}}\frac{\partial\kappa_2}{\partial u}\quad\text{and}\quad\frac{\partial\mathbf{\Phi}}{\partial v}=\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{u}}\frac{\partial\kappa_1}{\partial v}+\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{v}}\frac{\partial\kappa_2}{\partial v}\,. \tag{96}$$

Hence we find

$$\frac{\partial\mathbf{\Phi}}{\partial u}\times\frac{\partial\mathbf{\Phi}}{\partial v} \;=\; \left(\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{u}}\times\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{v}}\right)\left(\frac{\partial\kappa_1}{\partial u}\frac{\partial\kappa_2}{\partial v}-\frac{\partial\kappa_2}{\partial u}\frac{\partial\kappa_1}{\partial v}\right)=\left(\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{u}}\times\frac{\partial\hat{\mathbf{\Phi}}}{\partial\hat{v}}\right)\det J_{\boldsymbol{\kappa}}\,, \tag{97}$$

where $\det J_{\boldsymbol{\kappa}}$ is the determinant of the Jacobi matrix of $\boldsymbol{\kappa}(u,v)$. Therefore a reparametrisation is simply a substitution and leads to the same surface integral up to a possible sign factor depending on the sign of $\det J_{\boldsymbol{\kappa}}$. We now also realise that the sign of the surface integral is not uniquely defined in [32]. We could have taken the cross product the other way round and would therefore have obtained the opposite sign. Of course we want to choose the same direction for the normal vector $\frac{\partial\mathbf{\Phi}}{\partial u}\times\frac{\partial\mathbf{\Phi}}{\partial v}$ all along the surface but this still leaves us with two possible choices leading to two different signs for the surface integral. Whenever the surface has an obvious *inside* and *outside* then we always want to choose $d\mathbf{S}$ to point in the direction of the outside of the surface, this we call the *outward normal*. However, not every surface has an obvious outside and inside and in this case we just note that the definition of the surface integral [32] is ambiguous in the sign factor and in order to have a well-defined surface integral we need to give explicitly the direction of $d\mathbf{S}$.

Example 3.13 *The function* $\mathbf{\Phi}(r,\phi)=(r\cos\phi,r\sin\phi,r)$ *describes the surface of a cone* $\mathcal{S}_1$ *for* $0\leq r\leq 1$, $0\leq\phi\leq 2\pi$. *Find the surface element* $d\mathbf{S}$ *and calculate the flow of the vector field* $\mathbf{F}=(0,0,1)$ *through the surface of the cone. Compare this with the flow through the surface* $\mathcal{S}_2$ *described by* $x^2+y^2\leq 1$ *and* $z=1$ *with the normal pointing in negative* z *direction.*

Using the right hand rule it is clear that the outward surface element is given by $d\mathbf{S}=\frac{\partial\mathbf{\Phi}}{\partial\phi}\times\frac{\partial\mathbf{\Phi}}{\partial r}d(r,\phi)=(-r\sin\phi,r\cos\phi,0)^T\times(\cos\phi,\sin\phi,1)^T d(r,\phi)=(r\cos\phi,r\sin\phi,-r)^T d(r,\phi)$. *Therefore* $\int_{\mathcal{S}_1}\mathbf{F}.d\mathbf{S}=\int_0^1\int_0^{2\pi}(-r)d\phi dr=-\pi$.

The surface $\mathcal{S}_2$ is flat and has obviously the surface element $d\mathbf{S} = (0,0,-1)^T d(x,y)$. We therefore find $\int_{\mathcal{S}_2} \mathbf{F}.d\mathbf{S} = \int_{x^2+y^2\leq 1}(-1)d(x,y) = -\pi$ which is the same value as for surface $\mathcal{S}_1$ (why?).

Let us repeat this for the vector field $\mathbf{G} = (0,0,z)^T$: $\int_{\mathcal{S}_1} \mathbf{G}.d\mathbf{S} = \int_0^1 \int_0^{2\pi}(-r^2)d\phi dr = -\frac{2}{3}\pi$ whilst $\int_{\mathcal{S}_2} \mathbf{G}.d\mathbf{S}$ remains $\int_{x^2+y^2\leq 1}(-1)d(x,y) = -\pi$. We observe that in this case the integrals are different. We will understand later when we discuss the Divergence Theorem why these integrals match in the first case and why they don't match in the second case.

Now we know how to integrate scalar products of surface elements and vector fields over a surface. Just like for integrals along paths we are also interested in integrating just the magnitude of $d\mathbf{S}$. If we go through exactly the same methodology as before but look at sums $\sum_{ij} f(\xi_i,\eta_j)\|\mathbf{S}_{ij}\|$ instead we obtain an integral which for $f = 1$ obviously defines the area of the surface $\mathcal{S}$. Following exactly the same procedure as above we find:

34 Surface area integrals

The function $\mathbf{\Phi} : \mathcal{P} \to \mathbb{R}^3$ is a parametrisation of the surface $\mathcal{S}$ with parameter space $\mathcal{P} \subset \mathbb{R}^2$ and $f(x,y,z)$ is a real valued function. The surface area integral of f over $\mathcal{S}$ is

$$\int_{\mathcal{S}} f(\mathbf{x})dS \;=\; \lim_{\|\mathcal{Z}_u\|,\|\mathcal{Z}_v\|\to 0} \sum_{i,j} f(\mathbf{\Phi}(\xi_i,\eta_j)).\|\mathbf{S}_{ij}\|\,, \tag{98}$$

where ξ_i and η_j are in $[u_{i-1},u_i]$ or $[v_{j-1},v_j]$ respectively. The area of the surface $\mathcal{S}$ is defined as

$$\|\mathcal{S}\| \;:=\; \int_{\mathcal{S}} dS\,. \tag{99}$$

Completely analogously to our earlier considerations we find that this integral can be converted into a Riemann integral where $\frac{\partial\mathbf{\Phi}(u,v)}{\partial u}\times\frac{\partial\mathbf{\Phi}(u,v)}{\partial v}$ will now appear as $\left\|\frac{\partial\mathbf{\Phi}(u,v)}{\partial u}\times\frac{\partial\mathbf{\Phi}(u,v)}{\partial v}\right\|$.

35 Surface area integrals over differentiable surfaces

The surface $\mathcal{S}$ is parametrised by the function $\mathbf{\Phi}$ over the parameter space P then the surface area integral of the real valued function $f(x,y,z)$ over $\mathcal{S}$ is

$$\int_{\mathcal{S}} f(\mathbf{x})dS \;=\; \int_P f(\mathbf{\Phi}(u,v))\left\|\frac{\partial\mathbf{\Phi}(u,v)}{\partial u}\times\frac{\partial\mathbf{\Phi}(u,v)}{\partial v}\right\| d(u,v)\,. \tag{100}$$

In particular, the area of the surface $\mathcal{S}$ is given by

$$\|\mathcal{S}\| \;=\; \int_P \left\|\frac{\partial\mathbf{\Phi}(u,v)}{\partial u}\times\frac{\partial\mathbf{\Phi}(u,v)}{\partial v}\right\| d(u,v)\,. \tag{101}$$

For simplicity we use the notation

$$dS \;=\; \left\|\frac{\partial\mathbf{\Phi}(u,v)}{\partial u}\times\frac{\partial\mathbf{\Phi}(u,v)}{\partial v}\right\| d(u,v)\,. \tag{102}$$

It seems that the main difficulty of solving surface integrals is finding $d\mathbf{S}$ for a surface $\mathcal{S}$. Once we have found $d\mathbf{S}$ then the integration is reduced to solving two successive Riemann integrals. In Eq. 94 we have already given $d\mathbf{S}$ for the (open) upper hemisphere of radius R. The same result obviously generalizes to a whole sphere with radius R.

36 Surface element of a sphere
The surface element of the sphere $x^2+y^2+z^2=r^2$ parametrised by $\phi\in[0,2\pi]$ and $\theta\in[0,\pi]$ is given by

$$d\mathbf{S} \;=\; r^2\sin\theta\begin{pmatrix}\cos\phi\,\sin\theta\\ \sin\phi\,\sin\theta\\ \cos\theta\end{pmatrix}d\theta\,d\phi = r^2\sin\theta\mathbf{e}_r d\theta\,d\phi = r\sin\theta\mathbf{x}\,d\theta\,d\phi\,, \tag{103}$$

and the surface area element is

$$dS \;=\; r^2\sin\theta\,d\theta\,d\phi\,. \tag{104}$$

Example 3.14 *Find the surface area element dS of the paraboloid $z=x^2+y^2$, $0\le z\le 1$ and find its area.*

We parametrise the paraboloid using the function $\mathbf{\Phi}(r,\phi)=(r\cos\phi,r\sin\phi,r^2)^T$ with $0\le\phi\le 2\pi$ and $r\ge 0$. The outward surface element is given by $d\mathbf{S}=\frac{\partial\mathbf{\Phi}}{\partial\phi}\times\frac{\partial\mathbf{\Phi}}{\partial r}d(r,\phi)=(-r\sin\phi,r\cos\phi,0)^T\times(\cos\phi,\sin\phi,2r)^T=(2r^2\cos\phi,2r^2\sin\phi,-r)^T$. Taking the magnitude of $d\mathbf{S}$ we find for the surface area element $dS=r\sqrt{1+4r^2}d(r,\phi)$. Therefore the area is $\int_0^1\int_0^{2\pi}r\sqrt{1+4r^2}d\phi dr=\frac{\pi}{6}(5^{\frac{3}{2}}-1)$.

For completeness we will define surface integrals over real valued functions as well as surface area integrals over vector valued functions.

37 Surface integrals over real valued functions

The integral of the real valued function $f(x,y,z)$ over the surface $\mathcal{S}\subset\mathbb{R}^3$ parametrised by the function $\mathbf{\Phi}$ with parameter space $\mathcal{P}$ is defined by

$$\int_{\mathcal{S}} f(x,y,z)d\mathbf{S} \;:=\; \int_{\mathcal{P}} f(\mathbf{\Phi}(u,v))\left(\frac{\partial\boldsymbol{\phi}(u,v)}{\partial u}\times\frac{\partial\boldsymbol{\phi}(u,v)}{\partial v}\right)d(u,v)\,. \tag{105}$$

38 Surface area integrals over vector valued functions
The integral of the vector valued function $\mathbf{F}(x,y,z)$ over the surface $\mathcal{S}$ is defined by.

$$\int_{\mathcal{S}}\mathbf{F}(\mathbf{x})dS \;:=\; \begin{pmatrix}\int_{\mathcal{S}}F_1(\mathbf{\Phi}(u,v))dS\\ \int_{\mathcal{S}}F_2(\mathbf{\Phi}(u,v))dS\\ \int_{\mathcal{S}}F_3(\mathbf{\Phi}(u,v))dS\end{pmatrix}. \tag{106}$$

Examples

Example 3.15 *We will find the normal to the surface S of revolution given parametrically by*

$$\mathbf{\Phi}(\phi,z)=(R(z)\cos\phi,\;R(z)\sin\phi,z)\;.$$

The two variables z and ϕ parametrise the surface. Note that in cylindrical polar co-ordinates (ρ,ϕ,z) the surface is given by

$$\rho=(x^2+y^2)^{\frac{1}{2}}=R(z)\,.$$

We have

$$\begin{aligned}\frac{\partial \mathbf{\Phi}}{\partial \phi} &= (-R\sin\phi,\ R\cos\phi,\ 0)\,,\\ \frac{\partial \mathbf{\Phi}}{\partial z} &= (R'(z)\cos\phi,\ R'(z)\sin\phi,\ 1)\,.\end{aligned}$$

These two vectors span the tangent plane to S at the point with parameters ϕ, z, as can be seen from the Taylor series, taken to linear order:

$$\mathbf{\Phi}(\phi+\delta\phi, z+\delta z) \approx \mathbf{\Phi}(\phi, z) + \frac{\partial \mathbf{\Phi}}{\partial \phi}\delta\phi + \frac{\partial \mathbf{\Phi}}{\partial z}\delta z.$$

The normal direction to S is therefore given by

$$\frac{\partial \mathbf{\Phi}}{\partial \phi} \times \frac{\partial \mathbf{\Phi}}{\partial z} = (R\cos\phi, R\sin\phi, -RR')$$

and the unit normal is

$$\pm\frac{(\cos\phi, \sin\phi, -R')}{(1+R'^2)^{\frac{1}{2}}}\,.$$

Example 3.16 *We will work out the vector and scalar elements of area for three different surfaces.*

(i) Let $\mathbf{\Phi}(x,y) = (x, y, z(x,y))$. Then, using subscripts to denote partial derivatives of the function z,

$$\frac{\partial \mathbf{\Phi}}{\partial x} = (1, 0, z_x)\,, \qquad \frac{\partial \mathbf{\Phi}}{\partial y} = (0, 1, z_y)\,.$$

Thus

$$d\mathbf{S} = (-z_x, -z_y, 1)\,dx\,dy\,,$$

and

$$dS = (1 + z_x^2 + z_y^2)^{\frac{1}{2}} dx\,dy\ \geq dx\,dy\,.$$

(ii) Let $\mathbf{\Phi}(\theta,\phi) = (a\sin\theta\cos\phi,\ a\sin\theta\sin\phi,\ a\cos\theta)$, which represents the surface of sphere of radius a. Then

$$\frac{\partial \mathbf{\Phi}}{\partial \theta} = (a\cos\theta\cos\phi,\ a\cos\theta\sin\phi,\ -a\sin\theta)\,, \qquad \frac{\partial \mathbf{\Phi}}{\partial \phi} = (-a\sin\theta\sin\phi,\ a\sin\theta\cos\phi,\ 0)\,,$$

so that

$$d\mathbf{S} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,a^2\sin\theta\,d\theta d\phi = \hat{\mathbf{x}}\,a^2\sin\theta\,d\theta\,d\phi\,,$$

where $\hat{\mathbf{x}}$ is the unit vector in the radial direction, and $dS = a^2\sin\theta\,d\theta\,d\phi$.

(iii) Consider the surface of revolution specified by (compare example 2.12)

$$\mathbf{\Phi}(\phi, z) = (R(z)\cos\phi, R(z)\sin\phi, z)\,.$$

We have

$$\frac{\partial \mathbf{\Phi}}{\partial \phi} = (-R\sin\phi, R\cos\phi, 0)\,, \qquad \frac{\partial \mathbf{\Phi}}{\partial z} = (R'(z)\cos\phi, R'(z)\sin\phi, 1)\,.$$

Hence

$$\begin{aligned} d\mathbf{S} &= (R\cos\phi, R\sin\phi, -RR')\,d\phi\,dz\\ &= (\cos\phi, \sin\phi, -R')R\,d\phi\,dz\,,\\ dS &= (1+R'^2)^{\frac{1}{2}}R\,d\phi\,dz\,.\end{aligned}$$

Example 3.17 *We will calculate*

$$I = \int_{\mathcal{S}} xy\, dS\,,$$

where $\mathcal{S}$ *is the portion of the surface* $x + y + z = a$ *in* $x \geq 0$, $y \geq 0$, $z \geq 0$.

Parameterising the surface with x *and* y, *we have* $\mathbf{\Phi} = (x, y, a - x - y)$, *so*

$$d\mathbf{S} = (1, 0, -1) \times (0, 1, -1)\; dx\; dy = (1, 1, 1)\; dx\; dy$$

and

$$dS = \sqrt{3}\, dx\, dy \equiv \sec\alpha\, dx\, dy\ ,$$

where α *is the angle between the plane and the horizontal. This has an obvious geometric interpretation:* dx, dy *is the projection of* dS *into the* x-y *plane. Hence*

$$\begin{aligned} I &= \sqrt{3}\int_0^a \int_0^{a-x} xy\; dydx \\ &= \frac{\sqrt{3}}{2}\int_0^a x(a-x)^2 dx \\ &= \frac{a^4}{8\sqrt{3}}\,. \end{aligned}$$

Example 3.18 *We will calculate*

$$\mathbf{I} = \int_{\mathcal{S}} \mathbf{x}\; dS\ ,$$

where $\mathcal{S}$ *is the hemisphere* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. *This is obviously going to be easiest in spherical polar coordinates, so we write*

$$\mathbf{\Phi} = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta), \qquad dS = a^2\sin\theta\, d\theta\, d\phi\ .$$

Of course, we must use Cartesian axes for the integrand: we must not be tempted to set $\mathbf{\Phi} = (a, 0, 0)$ *in spherical polar axes, despite the simplification that would result. Thus*

$$\mathbf{I} = \int_0^{2\pi}\int_0^{\pi/2} (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)a^2\sin\theta\; d\theta\, d\phi.$$

We integrate each component separately. It is a good idea to do the ϕ *integral first, since some of the terms integrate to zero.*

$$\mathbf{I} = \int_0^{\pi/2} (0, 0, 2\pi a\cos\theta)a^2\sin\theta\; d\theta = (0, 0, \pi a^3).$$

Table of integrals

The following table summarises all integrals we have studied in the previous chapter and how to convert them into Riemann Integrals. $\boldsymbol{\gamma}(\mathbf{x})$ denotes a Jordan path defined on the interval $[a,b]$ and $\mathbf{F}(t)$ denotes the vector valued function $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix}$. The surface $\mathcal{S}$ is parametrised by the function $\boldsymbol{\Phi}(u,v)$ with parameter space $\mathcal{P}$.

Integral	notation	corresponding integral(s)
Integral along $\boldsymbol{\gamma}(t)$	$\int_{\boldsymbol{\gamma}} f(\mathbf{x})ds$	$= \int_a^b f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt$[1cm]
Length of $\boldsymbol{\gamma}(t)$	$\int_{\boldsymbol{\gamma}} ds$	$= \int_a^b \|\boldsymbol{\gamma}'(t)\|dt$
Integral along $\boldsymbol{\gamma}(t)$ of $\mathbf{F}$	$\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{x})ds$	$= \begin{pmatrix} \int_a^b f_1(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt \\ \vdots \\ \int_a^b f_m(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt \end{pmatrix}$
Line integrals	$\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{x}).d\mathbf{x}$	$= \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)).\boldsymbol{\gamma}'(t)dt$
Flat surface integral	$\int_{\mathcal{S}} f(x,y)dS$	$= \int_a^b \int_{c(x)}^{d(x)} f(x,y)dydx$
Area of a flat surface	$\int_{\mathcal{S}} dS$	$= \int_a^b \int_{c(x)}^{d(x)} dydx$
Volume integral	$\int_{\mathcal{V}} f(x,y,z)dV$	$= \int_a^b \int_{c(x)}^{d(x)} \int_{g(x,y)}^{h(x,y)} f(x,y,z)dzdydx$
Volume	$\int_{\mathcal{V}} dV$	$= \int_a^b \int_{c(x)}^{d(x)} \int_{g(x,y)}^{h(x,y)} dzdydx$
Flat surface integral of $\mathbf{F}$	$\int_{\mathcal{S}} \mathbf{F}(x,y)dS$	$= \begin{pmatrix} \int_a^b \int_{c(x)}^{d(x)} f_1(x,y)dydx \\ \vdots \\ \int_a^b \int_{c(x)}^{d(x)} f_m(x,y)dydx \end{pmatrix}$
Volume integral of $\mathbf{F}$	$\int_{\mathcal{V}} \mathbf{F}(x,y,z)dV$	$= \begin{pmatrix} \int_a^b \int_{c(x)}^{d(x)} \int_{g(x,y)}^{h(x,y)} f_1(x,y,z)dzdydx \\ \vdots \\ \int_a^b \int_{c(x)}^{d(x)} \int_{g(x,y)}^{h(x,y)} f_m(x,y,z)dzdydx \end{pmatrix}$
Surface integral over $\mathcal{S}$	$\int_{\mathcal{S}} \mathbf{F}(\mathbf{x}).d\mathbf{S}$	$= \int_{\mathcal{P}} \mathbf{F}(\boldsymbol{\Phi}(u,v)).\left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right) d(u,v)$
Surface area integral	$\int_{\mathcal{S}} f(\mathbf{x})dS$	$= \int_{\mathcal{P}} f(\boldsymbol{\Phi}(u,v))\left\|\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right\| d(u,v)$
Area of a surface	$\int_{\mathcal{S}} dS$	$= \int_{\mathcal{P}} \left\|\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right\| d(u,v)$
Surface area integral of $\mathbf{F}$	$\int_{\mathcal{S}} \mathbf{F}(\mathbf{x})dS$	$= \begin{pmatrix} \int_{\mathcal{P}} f_1(\boldsymbol{\Phi}(u,v))\left\|\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right\| d(u,v) \\ \vdots \\ \int_{\mathcal{P}} f_m(\boldsymbol{\Phi}(u,v))\left\|\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\right\| d(u,v) \end{pmatrix}$