Vector Calculus
Mathematical Tripos, Part IA
Lecture Notes

I INTRODUCTION AND NOTATION

I Introduction and Notation

Introduction, books, notation, important results from Part IA Differential Equations.

II Curves in $\mathbb{R}^3$

Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1 hour]

III Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4 hours]

IV Vector Operators

Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical and general orthogonal curvilinear coordinates; conservative fields. Divergence, curl and $\nabla^2$ in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical and general orthogonal curvilinear coordinates. Solenoidal fields and irrotational fields. Vector derivative identities. [5 hours]

V Integration Theorems

Green’s theorem in the plane, Divergence theorem, Stokes’ theorem, Green’s first and second theorem: statements; informal proofs; examples; application to fluid dynamics and electromagnetism. [5 hours]

VI Laplace’s Equation

Laplace’s equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson’s equation by Gauss’ method (for spherical and cylindrical symmetry) and as an integral. [4 hours]

VII Cartesian Tensors in $\mathbb{R}^3$

Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5 hours]
http://www.damtp.cam.ac.uk/user/md131/vector_calculus
md131@cam.ac.uk
Books

Notation

The following is a table of the notation used in this course.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Example</th>
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<tbody>
<tr>
<td>$\forall$</td>
<td>for all</td>
<td>$\forall x \in \mathbb{R}$</td>
</tr>
<tr>
<td>$\exists$</td>
<td>there exists</td>
<td>$\exists x \in \mathbb{C}$</td>
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<td>$\exists!$</td>
<td>there exists a unique</td>
<td>$\exists! x \in \mathbb{Z}$</td>
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<td>there is no</td>
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<tr>
<td>$\Rightarrow$</td>
<td>implication</td>
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</tr>
<tr>
<td>$\therefore$</td>
<td>therefore</td>
<td>$(a, b)$</td>
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<td>$::$</td>
<td>because</td>
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<td>$\leftrightarrow$</td>
<td>equivalence</td>
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<td>$:$</td>
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<td>$:$</td>
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<td>$\equiv$</td>
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<td>$\approx$</td>
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<td>$\downarrow$</td>
<td>contradiction</td>
<td>$x \downarrow y$</td>
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Cylindrical polar co-ordinates \((\rho, \phi, z)\) in \(\mathbb{R}^3\).

In cylindrical polar co-ordinates the position vector \(\mathbf{x}\) is given in terms of a radial distance \(\rho\) from an axis \(\mathbf{k}\), a polar angle \(\phi\), and the distance \(z\) along the axis. With respect to cartesian axes, the position vector is

\[
\mathbf{x} = (\rho \cos \phi, \rho \sin \phi, z),
\]

where \(0 \leq \rho < \infty\), \(0 \leq \phi \leq 2\pi\) and \(-\infty < z < \infty\).

In terms of the orthonormal basis related to the cylindrical coordinates, \(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\), the position vector is given by

\[
\mathbf{r} = \rho \mathbf{e}_\rho + z \mathbf{e}_z.
\]

Spherical polar co-ordinates \((r, \theta, \phi)\) in \(\mathbb{R}^3\).

In spherical polar co-ordinates the position vector \(\mathbf{x}\) is given in terms of a radial distance \(r\) from the origin, a ‘latitude’ angle \(\theta\), and a ‘longitude’ angle \(\phi\). With respect to cartesian axes, the position vector is

\[
\mathbf{x} = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta),
\]

where \(0 \leq r < \infty\), \(0 \leq \theta \leq \pi\) and \(0 \leq \phi \leq 2\pi\).

In terms of the orthonormal basis \(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\), the position vector is given by

\[
\mathbf{x} = r \mathbf{e}_r.
\]
Fig. iii Graph of $f(x,y) = e^{-(x^2 + y^2)}$
\[
\frac{d}{dt} f(x_1(t), x_2(t), \ldots, x_n(t)) = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt},
\]

(5)

\[
\left( \frac{\partial f}{\partial t} \right)_{x_{i_1}, \ldots, x_{i_k}} = \frac{\partial f}{\partial x_1} \left( \frac{\partial x_1}{\partial t} \right)_{x_{i_1}, \ldots, x_{i_k}} + \ldots + \frac{\partial f}{\partial x_n} \left( \frac{\partial x_n}{\partial t} \right)_{x_{i_1}, \ldots, x_{i_k}}.
\]

(6)
1 Schwarz's theorem

Mixed partial derivatives are independent of the order of differentiation e.g.:

\[
\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}.
\]  

*(Hermann Amandus Schwarz, 1843-1921).*
2 Increments of real valued functions:

\[
\begin{align*}
    f(x + h) - f(x) &= \frac{\partial f(x)}{\partial x_1} h_1 + \ldots + \frac{\partial f(x)}{\partial x_n} h_n + r(h) \\
    \text{with } \frac{r(h)}{\|h\|} &\to 0 \text{ as } h \to 0.
\end{align*}
\]
3 Taylor’s theorem

\[ f(x + h) = f(x) + \sum_i \frac{\partial f(x)}{\partial x_i} h_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} h_i h_j + \ldots \\
\ldots + \frac{1}{k!} \sum_{i_1, \ldots, i_k} \frac{\partial^k f(x)}{\partial x_{i_1} \ldots \partial x_{i_k}} h_{i_1} \ldots h_{i_k} + r_k(h), \tag{12} \]

with \( \frac{r_k(h)}{||h||} \to 0 \) as \( h \to 0 \).
\[ f(x + h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2} h^T H_f(x) h + \ldots, \] (14)

with

\[
H_f(x) = \begin{pmatrix}
\frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f(x)}{\partial x_n^2} & \frac{\partial^2 f(x)}{\partial x_n \partial x_{n-1}} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_1}
\end{pmatrix}.
\]
II Curves in $\mathbb{R}^3$

Fig. v A path $\gamma(t)$ in $\mathbb{R}^3$. 
Fig. vi Partitioning a path $\gamma(t)$ in $\mathbb{R}^3$. 
4 Length of a path

The limit $\lim_{Z \to 0} L(\gamma, Z)$ of a (rectifiable) path $\gamma(t) : [a, b] \to \mathbb{R}^n$ is called the length $L(\gamma)$ of the path $\gamma$ and is denoted by

$$\int_\gamma ds := \lim_{\|Z\| \to 0} L(\gamma, Z).$$  \hspace{1cm} (19)
5 Length of a differentiable path
If the path $\gamma : [a, b] \to \mathbb{R}^n$ can be differentiated, then the length of $\gamma$ is given by:

$$\int_{\gamma} ds = \int_{a}^{b} \|\gamma'(t)\|dt.$$  \hspace{1cm} (22)

We call $\gamma'(t)$ the velocity, $\|\gamma'(t)\|$ the speed and $\gamma''(t)$ the acceleration of the path $\gamma(t)$. 
Example 2.1 The path $\gamma(t) = (t \cos t, t \sin t)$ describes a spiral turning counterclockwise as $t$ runs from 0 to $2\pi n$, $n \in \mathbb{N}$. What is the length of the spiral?

We find the velocity vector as $\gamma' = (\cos t - t \sin t, \sin t + t \cos t)$ and therefore its norm (the speed) is $\|\gamma'\| = \sqrt{1 + t^2}$. Therefore $\int_0^{2\pi n} ds = \int_0^{2\pi n} \sqrt{1 + t^2} dt = \frac{1}{2}(t\sqrt{1 + t^2} + \sinh^{-1} t)\bigg|_0^{2\pi n} = \pi n \sqrt{1 + 4\pi^2 n^2} + \frac{1}{2} \sinh^{-1}(2\pi n)$. 
6. The sum of paths and the length function

If \( s(t) \) describes the length of the path \( \gamma \) restricted to the interval \([a, t]\) then \( s'(t) = \| \gamma'(t) \| \). If \( \gamma \) is the sum of finitely many paths \( \gamma = \gamma_1 \oplus \ldots \oplus \gamma_k \) then

\[
\int_{\gamma} ds = \int_{\gamma_1} ds + \ldots + \int_{\gamma_k} ds.
\]  

(25)
7 Arc length of a Jordan arc

The arc length of a Jordan arc \( \Gamma \) is the length of a Jordan path, an injective and rectifiable path \( \gamma \) describing \( \Gamma \). Two such Jordan paths describing \( \Gamma \) have the same length and are reparametrisations of one another. We denote the length of a Jordan arc by \( \int\gamma \, ds \).
\textbf{8 Length of a curve described by a function } f(x) \\

The length of a continuous curve \( \Gamma_f \) described by a function \( f(x) : [a, b] \to \mathbb{R} \) is

\[
\int_{\Gamma_f} ds = \int_a^b \sqrt{1 + \left( \frac{df}{dx} \right)^2} \, dx.
\]  

(28)
Fig. vii  Circular helix for $a = c = 1$. 
Integrals over $f(x)$ along paths

If the path $\gamma : [a, b] \to \mathbb{R}^n$ can be differentiated then:

$$\int_{\gamma} f(x) ds = \int_{a}^{b} f(\gamma(t)) \| \gamma'(t) \| dt . \tag{30}$$
10 Integration rules for integrals over \( f(x) \) along paths

\[
\int_{\gamma} (\alpha f(x) + \beta g(x))ds = \alpha \int_{\gamma} f(x)ds + \beta \int_{\gamma} g(x)ds ,
\]

\[
\int_{\gamma_1 \oplus \gamma_2} f(x)ds = \int_{\gamma_1} f(x)ds + \int_{\gamma_2} f(x)ds .
\]
\[
\int_{\gamma} f(\mathbf{x}) ds := \begin{pmatrix}
\int_{\gamma} f_1(\mathbf{x}) ds \\
\vdots \\
\int_{\gamma} f_m(\mathbf{x}) ds
\end{pmatrix}.
\]
1 Tangents and normals to curves in $\mathbb{R}^3$

11 Curvature of a plane curve

$\gamma_\epsilon(s)$ is the canonical Jordan path of the Jordan arc $\Gamma$. The curvature $\kappa(s)$ of $\Gamma$ at the position $s$ is defined as the magnitude of the acceleration vector

$$\kappa(s) := \| \gamma''(s) \|. \quad (34)$$
Fig. viii Osculating circle to a Jordan arc.
12 Curves in $\mathbb{R}^3$

Let $\Gamma$ be a Jordan arc in $\mathbb{R}^3$ and let $\gamma_c$ be the corresponding canonical Jordan path $\gamma_c : [a, b] \to \mathbb{R}^3$.

\[ t := \gamma'_c(s) \quad (36) \]

is called the tangent and it is clear that $\|t(s)\| = 1 \quad \forall s$.

\[ p := \frac{\gamma''_c(s)}{\|\gamma''_c(s)\|} \quad (37) \]

is called the principal normal and $\kappa(s) := \|\gamma''_c(s)\|$ is the curvature. The binormal is given by

\[ b := t \times p , \quad (38) \]

and the torsion $\tau$ is the (negative) projection of the change in the binormal $\frac{\partial b}{\partial s}$ onto the principal normal:

\[ \tau := -\frac{\partial b}{\partial s} \cdot p . \quad (39) \]
FIG. ix  Tangent, principal normal and binormal for a circular helix.
III Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

Fig. xiii The Riemann integral.
13 Riemann sum

A Riemann sum is a sum of the type

\[ \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \]  \hspace{1cm} (40)

where all \( \xi_i \in [x_{i-1}, x_i] \).
3 Line integrals

14 Vector fields\textsuperscript{a}

A vector valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a vector field.

\textsuperscript{a}Note that for a vector field the domain and the range have to have the same dimension.
Fig. xiv  Two dimensional vector field.
Fig. xv Three dimensional vector field.
15 Line integrals

Given a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, and a path $\gamma : [a, b] \to \mathbb{R}^n$. The line integral of $\mathbf{F}$ along the path $\gamma$ is defined

$$\int_{\gamma} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} := \lim_{\|\Delta\| \to 0} \sum \mathbf{F}(\gamma(t_i)). (\gamma(t_i) - \gamma(t_{i-1})) \ .$$

(413)

An alternative notation for the line integral is

$$\int_{\gamma} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_{\gamma} F_1(\mathbf{x})dx_1 + F_2(\mathbf{x})dx_2 + \ldots + F_n(\mathbf{x})dx_n \ .$$

(414)
16 Line integral over continuous vector fields along differentiable paths

\[ \gamma : [a, b] \rightarrow \mathbb{R}^n \] is a differentiable path then the line integral can be evaluated using a simple Riemann integral:

\[ \int_{\gamma} \mathbf{F}(\mathbf{x}) \, d\mathbf{x} = \int_{a}^{b} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \, . \] (47)

Using the alternative notation we can write Eq. 47 in the following form\(^b\):

\[ \int_{\gamma} F_1(x) \, dx_1 + \ldots + F_n(x) \, dx_n = \int_{a}^{b} F_1(\gamma(t)) \frac{d\gamma_1}{dt} \, dt + \ldots + \int_{a}^{b} F_n(\gamma(t)) \frac{d\gamma_n}{dt} \, dt \, . \]

---

\(^b\)Sometimes \( \int_{a}^{b} F_1(\gamma(t)) \frac{d\gamma_1}{dt} \, dt + \ldots \) is shortened to \( \int_{a}^{b} F \, dx_1 + \ldots \). Note the danger of this notation: \( \int F \, dx_1 \) still has to be taken along the path and does not mean that \( x_1, \ldots, x_n \) can be considered as independent variables for the integration.
Example 3.1 Let us consider the vector field \( \mathbf{F}(\mathbf{x}) := \left( \frac{\mathbf{x}}{r^2}, \frac{\mathbf{x}}{r^3}, \frac{\mathbf{x}}{r^4} \right)^T \) given in Fig. xv and the continuous path \( \gamma(t) = (\cos t, \sin t, t)^T \) along the circular helix shown in Fig. vii for \( t \in [0, 2\pi] \) we find

\[
\int_{\gamma} \mathbf{F} \, d\mathbf{x} = \int_0^{2\pi} \begin{pmatrix} \cos t \\ \frac{\sin t}{(1+t^2)^{3/2}} \\ \frac{t}{(1+t^2)^{3/2}} \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} dt = \int_0^{2\pi} \frac{t}{(1+t^2)^{3/2}} dt = 1 - \frac{1}{1+4\pi^2}.
\] (48)
17 Integration rules for line integrals

\[ \int_\gamma (\alpha F + \beta G) \, dx = \alpha \int_\gamma F \, dx + \beta \int_\gamma G \, dx, \]  
(49)

\[ \int_{\gamma_1 \cup \gamma_2} F \, dx = \int_{\gamma_1} F \, dx + \int_{\gamma_2} F \, dx, \]  
(50)

\[ \int_\gamma F \, dx = - \int_{\gamma^*} F \, dx \quad \text{with} \quad \gamma^*(t) = \gamma(b - t), \]  
(51)

\[ \left| \int_\gamma F \, dx \right| \leq \max_{t \in [a, b]} \| F(\gamma(t)) \| \int_\gamma ds. \]  
(52)
4 Plane surface integrals in $\mathbb{R}^2$ and volume integrals in $\mathbb{R}^3$

![Graph of $f(x, y)$ over the rectangle $R$.](image1)

![Partitioning the rectangle $R$.](image2)
18 Integrals over a rectangle
The real valued function $f(x,y)$ is defined on the rectangle $\mathcal{R} = [a,b] \times [c,d]$. The integral
\[
\int_{\mathcal{R}} f(x,y) \, d(x,y) := \lim_{\|z_x\|,\|z_y\| \to 0} \sum_{i,j} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) ,
\]
with $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$. Instead of $\int_{\mathcal{R}} f(x,y) \, d(x,y)$ we may also use the notation $\int_{\mathcal{R}} f(x,y) \, dS$. 
19 Fubini’s theorem in two dimensions

The real valued function $f(x,y)$ is defined on the rectangle $\mathcal{R} = [a,b] \times [c,d]$. If the integrals $\int_c^d f(x,y)dy$ exist for all $x \in [a,b]$ then

$$\int_{\mathcal{R}} f(x,y)d(x,y) = \int_a^b \int_c^d f(x,y)dy \, dx ,$$

and if the integrals $\int_a^b f(x,y)dx$ exist for all $y \in [c,d]$ then

$$\int_{\mathcal{R}} f(x,y)d(x,y) = \int_c^d \int_a^b f(x,y)dx \, dy .$$
Fig. xviii  Graph of \( f(x, y) = xy e^{-(x^2 + y^2)} \).
20 Integrals over 3-dimensional cuboids

The real valued function \( f(x, y, z) \) is defined on the cuboid \( Q = [a, b] \times [c, d] \times [e, f] \). The integral

\[
\int_Q f(x, y, z) d(x, y, z) := \lim_{\|z\|\|z\|\|z\| \to 0} \sum_{i,j,k} f(\xi_i, \eta_j, \mu_k) \Delta x \Delta y \Delta z,
\]

with \( \xi_i \in [x_{i-1}, x_i] \), \( \eta_j \in [y_{j-1}, y_j] \) and \( \mu_k \in [z_{k-1}, z_k] \). Instead of \( \int_Q f(x, y, z) d(x, y, z) \) we may also use the notation \( \int_Q f(x, y, z) dV \).
21 Fubini’s theorem

The real valued function \( f(x, y, z) \) is defined on the cuboid \( Q = [a, b] \times [c, d] \times [e, f] \). If the integrals \( \int_c^f \int_c^d f(x, y, z) \, dz \, dy \) and \( \int_c^f \int_c^d f(x, y, z) \, dz \, dx \) exist for all \( x \in [a, b] \) and all \( y \in [c, d] \), then

\[
\int_Q f(x, y, z) \, d(x, y, z) = \int_a^b \int_c^d \int_c^f f(x, y, z) \, dz \, dy \, dx.
\] (59)
22 Exchange of order of integration

The function \( f(x,y) \) is integrable on the rectangle \( R = [a,b] \times [c,d] \) then

\[
\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy = \int_R f(x,y) \, d(x,y),
\]

provided all inner integrals \( \int_a^b f(x,y) \, dx \) and \( \int_c^d f(x,y) \, dy \) exist for all \( y \in [c,d] \) and all \( x \in [a,b] \) respectively.
5 Integration over plane surfaces and Volumes

\[ \chi_B(\mathbf{x}) := \begin{cases} 
1 & , \mathbf{x} \in B \\
0 & , \mathbf{x} \notin B 
\end{cases} \]  

(61)
23 Integrals over plane surfaces
The integral of the real valued function \( f(x, y) \) over the set \( S \) is:

\[
\int_S f(x, y) \, dS := \int_{\mathcal{R}} f(x, y) \chi_S(x, y) \, dS,
\]

(62)

where \( \mathcal{R} \) is a rectangle completely containing \( S \). In particular

\[
\|S\| := \int_S dS,
\]

(63)

is called the area of \( S \).

*Instead of \( \int_S f(x, y) \, dS \) we may also frequently use the notations \( \iint_S f(x, y) \, d(x, y) \).
24 Integrals over volumes

The integral of the real valued function \( f(x, y, z) \) over the set \( \mathcal{V} \) is\(^d\):

\[
\int_{\mathcal{V}} f(x, y, z) dV := \int_{\mathcal{Q}} f(x, y, z) \chi_{\mathcal{V}}(x, y, z) dV,
\]

where \( \mathcal{Q} \) is a cuboid completely containing \( \mathcal{V} \). In particular

\[
\|\mathcal{V}\| := \int_{\mathcal{V}} dV,
\]

is called the volume of \( \mathcal{V} \).

\(^d\)Instead of \( \int_{\mathcal{V}} f(x, y, z) dV \) we may also frequently use the notation \( \int_{\mathcal{V}} f(x, y, z) d(x, y, z) \).
25 Integration rules

\[
\int_{V} (\alpha f(\mathbf{x}) + \beta g(\mathbf{x})) dV = \alpha \int_{V} f(\mathbf{x}) dV + \beta \int_{V} g(\mathbf{x}) dV ,
\]

(66)

\[
\int_{V_{1} \cup V_{2}} f(\mathbf{x}) dV = \int_{V_{1}} f(\mathbf{x}) dV + \int_{V_{2}} f(\mathbf{x}) dV ,
\]

(67)

where \( V_{1} \) and \( V_{2} \) are disjoint or at least disjoint up to boundary points. Similar rules apply for plane surface integrals.
Fig. xix $S$ bounded by $x^2 + y^2 = 1; x, y \geq 0.$
26 Integrals over vector valued functions

The integral of the vector valued function $f(\mathbf{x})$ over the volume $\mathcal{V}$ is:

$$\int_{\mathcal{V}} f(\mathbf{x}) dV := \begin{pmatrix}
\int_{\mathcal{V}} f_1(\mathbf{x}) dV \\
\vdots \\
\int_{\mathcal{B}} f_m(\mathbf{x}) dV
\end{pmatrix}, \quad (69)$$

and similarly for plane surface integrals.
6 Substitution rule

27 Substitution rule for integrals in $\mathbb{R}^n$
If $g : B \rightarrow \hat{B}$ is a bijective transformation from the set $B$ onto $\hat{B} = g(B)$ with either $\det J_g(t) > 0$ or $\det J_g(t) < 0$ for all $t \in B$ then

$$\int_{g(B)} f(x) d(x_1, \ldots, x_n) = \int_B f(g(t)) |\det J_g(t)| d(t_1, \ldots, t_n),$$

(71)

where $|\det J_g(t)|$ denotes the absolute value of the determinant of the Jacobi matrix of the transformation $g(t)$. $|\det J_g(t)|$ is called the Jacobian of the transformation $g(t)$.

We call the function $g(t)$ a substitution function and for the Jacobian we use the notation

$$\frac{d(x_1, \ldots, x_n)}{d(t_1, \ldots, t_n)} := |\det J_g(t)|.$$

(72)
28 Transformation to plane polar coordinates

The Jacobian for plane polar coordinates is

\[ \frac{d(x, y)}{d(r, \phi)} = r. \]  \hspace{1cm} (75)

The transformation to plane polar coordinates is then given by

\[ \int_{\hat{S}} f(x, y) d(x, y) = \int_{S} f(r \cos \phi, r \sin \phi) r \, d(r, \phi), \]  \hspace{1cm} (76)

where \( \hat{S} \) and \( S \) describe the corresponding surface in the \( xy \)-plane or in the \( r\phi \)-plane respectively.
29 Transformation to cylindrical polar coordinates

The Jacobian for cylindrical polar coordinates is

$$\frac{d(x, y, z)}{d(\rho, \phi, z)} = \rho.$$  \hspace{1cm} (77)

The transformation to cylindrical polar coordinates is then given by

$$\int_{\mathcal{V}} f(x, y, z) d(x, y, z) = \int_{\hat{\mathcal{V}}} f(\rho \cos \phi, \rho \sin \phi, z) \rho \, d(\rho, \phi, z),$$  \hspace{1cm} (78)

where $\hat{\mathcal{V}}$ and $\mathcal{V}$ describe the corresponding volume in $xyz$-space or in $\rho\phi z$-space respectively.
Transformation to spherical polar coordinates

The Jacobian for spherical polar coordinates is

\[
\frac{d(x, y, z)}{d(r, \phi, \theta)} = r^2 \sin \theta. \tag{79}
\]

The transformation to spherical polar coordinates is then given by

\[
\int_{\hat{V}} f(x, y, z) d(x, y, z) = \int_{V} f(r \cos \phi \sin \theta, r \sin \phi \sin \theta, r \cos \theta) \ r^2 \sin \theta \ d(r, \phi, \theta), \tag{80}
\]

where \(\hat{V}\) and \(V\) describe the corresponding volume in \(xyz\)-space or in \(r\phi\theta\)-space respectively.
31 Jacobian of the inverse transformation

If \( g : \mathcal{B} \to \mathcal{B} \) is a substitution function then \( g(t) \) is invertible and the inverse function \( g^{-1} : \mathcal{B} \to \mathcal{B} \) is again a substitution function with Jacobian

\[
|\det J_{g^{-1}}(x)| = \frac{1}{|\det J_g(t)|},
\]

or in other words:

\[
\frac{d(x_1, \ldots, x_n)}{d(t_1, \ldots, t_n)} = \frac{1}{\frac{d(t_1, \ldots, t_n)}{d(x_1, \ldots, x_n)}}.
\]
7 Surface integrals in $\mathbb{R}^3$

**Fig. xx** Flow through a surface $S$. 
32 Surface integrals over vector fields

The function \( \Phi : \mathcal{P} \rightarrow \mathbb{R}^3 \) is a parametrisation of the surface \( S \) with parameter space \( \mathcal{P} \subset \mathbb{R}^2 \) and \( \mathbf{F} \) is a vector field in \( \mathbb{R}^3 \). The surface integral of \( \mathbf{F} \) over the surface \( S \) is

\[
\int_S \mathbf{F}(\mathbf{x}) \, dS = \lim_{\|z_n\|,\|z_n\| \to 0} \sum_{i,j} \mathbf{F}(\Phi(\xi_i, \eta_j)) \cdot \mathbf{s}_{ij} ,
\]

(86)

where \( \xi_i \) and \( \eta_j \) are in \([u_{i-1}, u_i]\) or \([v_{j-1}, v_j]\) respectively.
33 Surface integrals over differentiable surfaces
The surface $S$ is parametrised by the function $\Phi$ on the parameter space $P$, then the surface integral of the vector field $\mathbf{F}$ over $S$ is

$$\int_S \mathbf{F}(\mathbf{x}) \, d\mathbf{S} = \int_P \mathbf{F}(\Phi(u,v)) \cdot \left( \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \right) \, d(u,v).$$  \hspace{1cm} (91)

For simplicity we use the notation

$$d\mathbf{S} = \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \, d(u,v).$$  \hspace{1cm} (92)
**Example 3.12** Let us for example consider the vector field \( \mathbf{F}(x) = \frac{x}{r^2} \). We assume that \( \mathbf{F}(x) \) describes the flow density of a liquid. How much of the liquid flows through the (open) upper hemisphere \( x^2 + y^2 + z^2 = R^2, \ z \geq 0 \)? In order to find \( dS \) for the upper hemisphere we parametrise it using

\[
\Phi(x) = \begin{pmatrix}
R \cos \phi \sin \theta \\
R \sin \phi \sin \theta \\
R \cos \theta
\end{pmatrix}.
\]

(91)

Taking the cross product of the two tangent vectors leads to

\[
dS = \frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} = \begin{pmatrix}
R \cos \phi \cos \theta \\
R \sin \phi \cos \theta \\
-R \sin \phi
\end{pmatrix} \times \begin{pmatrix}
-R \sin \phi \sin \theta \\
R \cos \phi \sin \theta \\
0
\end{pmatrix} = R^2 \sin \theta e_r, d\theta d\phi.
\]

(92)

We therefore obtain for the flow of the liquid through the upper hemisphere

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \frac{\vec{r}}{R^3} R^2 \sin \theta \cdot e_r, d\theta d\phi = \int_0^{2\pi} \int_0^\pi \vec{r} \sin \theta d\theta d\phi = 2\pi.
\]

(93)
Example 3.13 The function $\Phi(r, \phi) = (r \cos \phi, r \sin \phi, r)$ describes the surface of a cone $S_1$ for $0 \leq r \leq 1$, $0 \leq \phi \leq 2\pi$. Find the surface element $dS$ and calculate the flow of the vector field $\mathbf{F} = (0,0,1)$ through the surface of the cone. Compare this with the flow through the surface $S_2$ described by $x^2 + y^2 \leq 1$ and $z = 1$ with the normal pointing in negative $z$ direction.

Using the right hand rule it is clear that the outward surface element is given by $dS = \frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial r} d(r,\phi) = (-r \sin \phi, r \cos \phi, 0)^T \times (\cos \phi, \sin \phi, 1)^T d(r,\phi) = (r \cos \phi, r \sin \phi, -r)^T d(r,\phi)$. Therefore $\int_{S_1} \mathbf{F} \cdot dS = \int_0^1 \int_0^{2\pi} (-r) d\phi dr = -\pi$.

The surface $S_2$ is flat and has obviously the surface element $dS = (0,0,-1)^T d(x,y)$. We therefore find $\int_{S_2} \mathbf{F} \cdot dS = \int_{x^2+y^2\leq 1} (-1)d(x,y) = -\pi$ which is the same value as for surface $S_1$ (why?).

Let us repeat this for the vector field $\mathbf{G} = (0,0,z)^T$: $\int_{S_1} \mathbf{G} \cdot dS = \int_0^1 \int_0^{2\pi} (-r^2) d\phi dr = -\frac{2\pi}{3}$. whilst $\int_{S_2} \mathbf{G} \cdot dS$ remains $\int_{x^2+y^2\leq 1} (-1)d(x,y) = -\pi$. We observe that in this case the integrals are different. We will understand later when we discuss the Divergence Theorem why these integrals match in the first case and why they don’t match in the second case.
34 Surface area integrals

The function $\Phi : \mathcal{P} \to \mathbb{R}^3$ is a parametrisation of the surface $S$ with parameter space $\mathcal{P} \subset \mathbb{R}^2$ and $f(x,y,z)$ is a real valued function. The surface area integral of $f$ over $S$ is

$$
\int_S f(\mathbf{x}) dS = \lim_{\|\mathbf{z}\|_i, \|\mathbf{z}\|_j \to 0} \sum_{i,j} f(\Phi(\xi_i, \eta_j)) \cdot \|S_{ij}\|, \tag{98}
$$

where $\xi_i$ and $\eta_j$ are in $[u_{i-1}, u_i]$ or $[v_{j-1}, v_j]$ respectively. The area of the surface $S$ is defined as

$$
\|S\| := \int_S dS. \tag{99}
$$
35 Surface area integrals over differentiable surfaces

The surface $S$ is parametrised by the function $\Phi$ over the parameter space $P$ then the surface area integral of the real valued function $f(x,y,z)$ over $S$ is

$$
\int_S f(x) dS = \int_P f(\Phi(u,v)) \left\| \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \right\| d(u,v).
$$

(100)

In particular, the area of the surface $S$ is given by

$$
\|S\| = \int_P \left\| \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \right\| d(u,v).
$$

(101)

For simplicity we use the notation

$$
dS = \left\| \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \right\| d(u,v).
$$

(102)
36 Surface element of a sphere
The surface element of the sphere \( x^2 + y^2 + z^2 = r^2 \) parametrised by \( \phi \in [0, 2\pi] \) and \( \theta \in [0, \pi] \) is given by

\[
dS = r^2 \sin \theta \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix} \, d\theta \, d\phi = r^2 \sin \theta e_r d\theta \, d\phi = r \sin \theta \mathbf{x} \, d\theta \, d\phi, \tag{103}
\]

and the surface area element is

\[
dS = r^2 \sin \theta \, d\theta \, d\phi. \tag{104}
\]
Example 3.14 Find the surface area element $dS$ of the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 1$ and find its area.

We parametrise the paraboloid using the function $\Phi(r, \phi) = (r \cos \phi, r \sin \phi, r^2)^T$ with $0 \leq \phi \leq 2\pi$ and $r \geq 0$. The outward surface element is given by $dS = \frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial r} dr d\phi = (-r \sin \phi, r \cos \phi, 0)^T \times (\cos \phi, \sin \phi, 2r)^T = (2r^2 \cos \phi, 2r^2 \sin \phi, -r)^T$. Taking the magnitude of $dS$ we find for the surface area element $dS = r \sqrt{1 + 4r^2} dr d\phi$. Therefore the area is

$$\int_0^1 \int_0^{2\pi} r \sqrt{1 + 4r^2} dr d\phi = \frac{\pi}{6} (5^2 - 1).$$
37 Surface integrals over real valued functions

The integral of the real valued function \( f(x,y,z) \) over the surface \( S \subset \mathbb{R}^3 \) parametrised by the function \( \Phi \) with parameter space \( P \) is defined by

\[
\int_S f(x,y,z) \, dS := \int_P f(\Phi(u,v)) \left( \frac{\partial \Phi(u,v)}{\partial u} \times \frac{\partial \Phi(u,v)}{\partial v} \right) \, d(u,v) .
\]  (105)
Surface area integrals over vector valued functions

The integral of the vector valued function \( \mathbf{F}(x, y, z) \) over the surface \( S \) is defined by.

\[
\int_S \mathbf{F}(\mathbf{x}) dS = \left( \int_S F_1(\Phi(u, v)) dS \right)
\left. \begin{array}{c}
\int_S F_2(\Phi(u, v)) dS \\
\int_S F_3(\Phi(u, v)) dS \\
\end{array} \right).
\] (106)
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8 Directional derivatives

39 Directional derivative
The directional derivative of the real valued function $f(x)$ is given by the limit

$$\frac{\partial f}{\partial n} := \lim_{h \to 0} \frac{f(x + hn) - f(x)}{h}$$
40 Directional derivatives theorem

\[ \frac{\partial f(x)}{\partial n} = \nabla f(x) \cdot n. \] (109)
41 Steepest increase and steepest decrease

For a real valued function \( f(x) \) we find that \( \nabla f(x) \) points in the direction of the steepest increase and \( -\nabla f(x) \) points in the direction of the steepest decrease.
9 Conservative fields

42 Conservative fields
A vector field $\mathbf{F}$ defined on the connected domain $G$ is called a conservative field if the line integrals $\int_{\gamma} \mathbf{F} \, dx$ only depend on the starting points and end points of $\gamma$ and are otherwise path independent. We can therefore denote $\int_{\gamma} \mathbf{F} \, dx$ simply by $\int_{\infty} \mathbf{F} \, dx$. 
43 Gradient fields\footnote{Note that for the definition of a gradient field we do not need to assume that the domain $G$ is connected but when we later ask the question whether the field is conservative we need to require that $G$ is connected since talking about conservative fields only makes sense if we are able to connect any two points with continuous paths.}

A vector field $\mathbf{F}$ is called a gradient field on the set $G$ if there exists a real valued function $\Phi(x)$ (called a scalar field) such that $\nabla \Phi(x) = \mathbf{F}(x) \ \forall x \in G$. $\Phi(x)$ is then called a potential\footnote{Note that physicists tend to call $-\Phi(x)$ with $\nabla \Phi(x) = -\mathbf{F}$ a potential.} for the vector field $\mathbf{F}(x)$.
Example 4.1 The scalar field $\Phi(x) = -\frac{1}{r}$ is a potential for the conservative field given in Fig. xv. Therefore this vector field is a gradient field on the set $\mathbb{R}^3 \setminus \{0\}$ as can easily be shown by verifying $\nabla \Phi(x) = F(x)$ $\forall x \in \mathbb{R}^3 \setminus \{0\}$. A two dimensional example is given by the vector field

$$F := \left( \begin{array}{c} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{array} \right).$$

(110)

It is easy to see that $\Phi(x,y) = \tan^{-1} \frac{y}{x}$ is a potential for this $F$ on the set $G = \mathbb{R}^2 \setminus \{(x,y) : x = 0\}$. Therefore $F$ is a gradient field on the set $G$. Note that $G$ is obviously not connected since it consists of two disconnected parts: the left half plane and the right half plane excluding the $y$-axis. However, $F$ is of course also a gradient field on the smaller set $\{(x,y) : x > 0\}$ which is connected. $\tilde{\Phi}(x,y) = -\tan^{-1} \frac{x}{y}$ is another potential for $F$ this time defined on the set $\mathbb{R}^2 \setminus \{(x,y) : y = 0\}$. Again this set is not connected. In fact, we shall see later that $F$ is not a gradient field on the whole of $\mathbb{R}^2 \setminus \{0\}$!
44 Line integrals over gradient fields
Assume that $\mathbf{F} : G \subset \mathbb{R}^n \to \mathbb{R}^n$, $G$ connected, is a gradient field with potential $\Phi(x)$ then $\mathbf{F}$ is also conservative on $G$ and the line integral along a path $\gamma(t) \subset G$ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma \nabla \Phi \cdot d\mathbf{x} = \Phi(x_e) - \Phi(x_s),$$

where $x_s = \gamma(b)$ is the end point and $x_e = \gamma(a)$ is the starting point of the path $\gamma$. 
45 Closed path integrals

A vector field $\mathbf{F} : G \subset \mathbb{R}^n \to \mathbb{R}^n$, $G$ connected, is conservative if and only if the closed line integrals $\oint \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed (rectifiable) paths $\gamma$ in $G$. 
46 Path independent line integrals

Assume that the vector field \( \mathbf{F} : G \subset \mathbb{R}^n \to \mathbb{R}^n \), \( G \) connected, is conservative, then \( \mathbf{F}(\mathbf{x}) \) is also a gradient field and

\[
\Phi(\mathbf{x}) := \int_{\mathbf{x}_a}^{\mathbf{x}} \mathbf{F} \, d\mathbf{x},
\]

is a potential for \( \mathbf{F} : \nabla \Phi(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \).
10 Integrability condition for vector fields

Fig. xxii A simply connected set.

Fig. xxiii A connected set which is not simply connected.
47 Integrability condition for vector fields

The vector field \( \mathbf{F} : G \subseteq \mathbb{R}^n \to \mathbb{R}^n \), \( G \) simply connected, is a gradient field and therefore also a conservative field if and only if the integrability condition

\[
\frac{\partial F_i(x)}{\partial x_j} = \frac{\partial F_j(x)}{\partial x_i}, \quad \forall x \in G \text{ and } \forall i, j,
\]

(113)

holds.
Example 4.2 The vector field $\mathbf{F} = \left(2x \sin(zy), x^2 z \cos(zy) + 2zy, x^2 y \cos(zy) + y^2 \right)^T$ is defined on the whole of $\mathbb{R}^3$ which is simply connected. It can easily be shown that $\nabla \times \mathbf{F} = 0$ and therefore $\mathbf{F}$ is indeed a gradient field. That means there exists a function $\Phi(\mathbf{x})$ such that $\frac{\partial \Phi}{\partial x} = 2x \sin(zy)$.

We can easily integrate this and obtain $\Phi(\mathbf{x}) = x^2 \sin(zy) + h(y, z)$ where $h(y, z)$ can only be a function of $y$ and $z$ but not of $x$. Therefore $\frac{\partial \Phi}{\partial y} = x^2 z \cos(zy) + \frac{\partial h(y, z)}{\partial y}$. Comparing this with the second component of the vector field $\frac{\partial \Phi}{\partial y} = x^2 z \cos(zy) + 2zy$ leads to $\frac{\partial h(y, z)}{\partial y} = 2zy$ which is indeed a relation without $x$ as required. We find $h(y, z) = zy^2 + g(z)$ with a function $g(z)$ which cannot be a function of $x$ and neither of $y$. Putting all this together we obtain $\frac{\partial \Phi}{\partial z} = x^2 y \cos(zy) + y^2$. This has to be compared to the third component $x^2 y \cos(zy) + y^2$ which gives an exact match and therefore $g(z) = \text{constant}$. We have therefore found the most general potential for $\mathbf{F}$ to be $\Phi(\mathbf{x}) = x^2 \sin(zy) + zy^2 + c$ with $c \in \mathbb{R}$. 
11 Differentials
48 Exact differentials

The differential \( F_1 \, dx_1 + \ldots + F_n \, dx_n \) is called an exact differential if and only if the vector field \( \mathbf{F} = (F_1, \ldots, F_n) \) is conservative. We then write \( d\Phi = F_1 \, dx_1 + \ldots + F_n \, dx_n \) with \( \nabla \Phi = \mathbf{F} \).
Example 4.3 For the differential $2xy\,dx + x^2\,dy$ we find that $\frac{\partial^2}{\partial x^2} = 2x = \frac{\partial^2}{\partial y^2}$ and therefore $2xy\,dx + x^2\,dy$ is exact. Trying to solve for example the differential equation $2xy + x^2y' = 0$ leads to $d(x^2y) = 0$ and therefore to the family of solutions $x^2y = \text{constant}$. 
49 Integrating factors for differentials
A real valued function $\lambda(x)$ is called an integrating factor for the differential $F_1 dx_1 + \ldots + F_n dx_n$ if the differential $\lambda(x) F_1 dx_1 + \ldots + \lambda(x) F_n dx_n$ is exact. Provided the differential and the integrating factor are both defined on a common open set $G$ which is simply connected then the integrability can be tested using

$$\frac{\partial \lambda(x) F_i}{\partial x_j} = \frac{\partial \lambda(x) F_j}{\partial x_i} \quad \forall i, j.$$  (120)
Example 4.4 It is obvious that the differential $2ydx + xdy$ is not exact since $\frac{\partial P}{\partial y} = 2 \neq 1 = \frac{\partial Q}{\partial x}$.

We try to find $\lambda(x,y)$ such that $\frac{\partial^2 \lambda}{\partial y^2} = \frac{\partial \lambda}{\partial x}$ which leads to a (partial) differential equation $2y \frac{\partial \lambda}{\partial y} + 2\lambda = x \frac{\partial \lambda}{\partial x} + \lambda$. Since we usually only need to find one integrating factor and not all integrating factors it is always worth trying if there exists an integrating factor which is only a function of $x$ (or only a function of $y$) such that $\frac{\partial \lambda}{\partial y} = 0$ (or $\frac{\partial \lambda}{\partial x} = 0$). The first assumption leads to $2\lambda = x\lambda'(x) + \lambda$ and therefore $\lambda = x$ is such a solution and is indeed not dependent on $y$ (note that all factors of $y$ disappeared once we had put $\frac{\partial \lambda}{\partial y} = 0$). Hence $2yxdx + x^2dy$ is exact.
Example 4.5 If we are given a differential of the form \( dy + (f(x)y - g(x))dx \) with some functions \( f(x) \) and \( g(x) \) then an integrating factor has to satisfy \( \frac{\partial \lambda}{\partial x} = \frac{\partial (f(x)y - g(x))}{\partial y} \frac{\lambda}{\partial y} \). Let us again try if we find an integrating factor \( \lambda(x) \) which depends only on \( x \) but not on \( y \). Therefore \( \lambda \) would have to satisfy \( \frac{\partial \lambda}{\partial x} = f(x)\lambda \) and therefore \( \lambda(x) = e^\int f(x)dx \) is an integrating factor such that \( \lambda(dy + (f(x)y - g(x))dx) = \lambda\lambda(x)y - \int \lambda(x)g(x)dx \) is exact. This is of course the standard integrating factor known to us for solving first order linear differential equations: \( y' + f(x)y = g(x) \). This differential equation can then be written as \( (\lambda(x)y)' = \lambda(x)g(x) \) with the \( \lambda(x) \) given above.
12 Surfaces in $\mathbb{R}^3$
50 Normal vectors to surfaces in $\mathbb{R}^3$

The vector $\nabla \Phi(x, y, z)$ is normal to the surface described by $\Phi(x, y, z) = 0$. 

Example 4.6 We will find the vector field normal to the family of surfaces given by

\[ \Phi(x, y, z) \equiv x^2 - y^2 - z^2 - c^2 = 0 \]

where \( c \) is a constant. Since the vector field \( \nabla \Phi \) is normal to the surfaces \( \Phi(x) = 0 \), and

\[ \nabla \Phi = 2(x, -y, -z), \]

the unit normal vector field is given by

\[ n = \pm(x, -y, -z)/r. \]

Note that the normal is not defined at \( r = 0 \). This is as expected, because the surface that passes through \( x = 0 \) has \( c = 0 \). It is the cone \( x^2 = y^2 + z^2 \), which does not have a normal at its apex.
13 Surface elements in $\mathbb{R}^3$
51 Surface elements in cartesian co-ordinates

The surface $S$ is defined by the equation $\Phi(x, y, z) = 0$. Provided that $\frac{\partial \Phi(x, y, z)}{\partial z} \neq 0$ we find for the surface element $dS$ in cartesian co-ordinates

$$dS = \frac{1}{\frac{\partial \Phi}{\partial z}} \nabla \Phi(x, y, z) d(x, y).$$

(122)

The surface area element is given by

$$dS = \frac{\| \nabla \Phi \|}{\frac{\partial \Phi}{\partial z}} d(x, y).$$

(123)

*Note that $dS$ in Eq. 122 is only defined up to a sign factor.*
Example 4.7 Using [51] we immediately find for the paraboloid described by \( z = x^2 + y^2 \) the surface element \( d\mathbf{S} = \left( \begin{array}{c} 2x \\ 2y \\ -1 \end{array} \right) \text{d}(x,y) \). The negative \( z \) co-ordinate of \( d\mathbf{S} \) shows that this is the outward normal as can easily be seen from Fig. xxix.

Applying [51] to the sphere \( x^2 + y^2 + z^2 = r^2 \) requires caution. For the hemisphere (or parts of the hemisphere) with \( z > 0 \) we easily find \( d\mathbf{S} = \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) \text{d}(x,y) \) and the positive \( z \) co-ordinate shows that it is the outward normal. However if we want to find \( d\mathbf{S} \) for the whole sphere or patches of the sphere which involve points with \( z = 0 \) this \( d\mathbf{S} \) would obviously not work. But using spherical polar co-ordinates and noting that \( d(x,y) = r^2 \sin \theta \cos \theta \text{d}(\phi, \theta) \) we can easily transform this \( d\mathbf{S} \) into \( r^2 \sin \theta \left( \begin{array}{c} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{array} \right) \text{d}(\phi, \theta) \) which is now valid everywhere on the sphere. It should also be noted that if we want to use \( d\mathbf{S} = \left( \begin{array}{c} x \\ y \\ 1 \end{array} \right) \text{d}(x,y) \) to integrate over the whole sphere and provided the limit \( z \to 0 \) works we would still need to integrate over both \( d\mathbf{S} = \left( \begin{array}{c} \frac{x}{r} \\ \frac{y}{r} \\ 1 \end{array} \right) \text{d}(x,y) \) over the disk with radius \( r \) to integrate along the upper hemisphere and \( d\mathbf{S} = \left( \begin{array}{c} \frac{x}{r} \\ \frac{y}{r} \\ -1 \end{array} \right) \text{d}(x,y) \) over the disk with radius \( r \) to integrate along the lower hemisphere in order to integrate over the whole sphere.
**Sphere** \( x^2 + y^2 + z^2 = r^2 \)

\[
dS = r^2 \sin \theta \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} d\theta d\phi ,
\]

\[
dS = r^2 \sin \theta d\theta d\phi ,
\]

\( \phi \in [0, 2\pi], \theta \in [0, \pi] . \)

**Cylinder** \( x^2 + y^2 = \rho^2, \ a \leq z \leq b \)

\[
dS = \rho \begin{pmatrix} \cos \phi & 0 \\ \sin \phi & 0 \end{pmatrix} dz d\phi ,
\]

\[
dS = \rho dz d\phi ,
\]

\( \phi \in [0, 2\pi], \ z \in [a, b] . \)

**Ellipsoid** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

\[
dS = abc \sin \theta \begin{pmatrix} \cos \phi \sin \theta & \sin \phi \sin \theta \\ \sin \phi \sin \theta & \cos \theta \end{pmatrix} d\theta d\phi .
\]

\( \phi \in [0, 2\pi], \ \theta \in [0, \pi] . \)

**Cone** \( x^2 + y^2 = z^2, \ 0 \leq z \leq b \)

\[
dS = r \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} dr d\phi ,
\]

\[
dS = \sqrt{2} r dr d\phi ,
\]

\( \phi \in [0, 2\pi], \ r \in [0, b] . \)
Hyperboloid \( x^2 + y^2 = z^2 + 1, \ 0 < a \leq z \leq b \)

\[
dS = \frac{r}{\sqrt{r^2 - 1}} \left( \begin{array}{c} r \cos \phi \\ r \sin \phi \\ -\sqrt{r^2 - 1} \end{array} \right) dr \, d\phi ,
\]

\[
dS = \frac{\sqrt{2r^2 - 1}}{\sqrt{r^2 - 1}} rd\,\phi ,
\]

\( \phi \in [0, 2\pi], \ r \in [\sqrt{a^2 + 1}, \sqrt{b^2 + 1}] \).

Paraboloid \( x^2 + y^2 = z, \ 0 \leq z \leq b \)

\[
dS = \left( \begin{array}{c} 2x \\ 2y \\ -1 \end{array} \right) dx \, dy = \left( \begin{array}{c} 2r \cos \phi \\ 2r \sin \phi \\ -1 \end{array} \right) rrd\,\phi ,
\]

\[
dS = \sqrt{1 + 4r^2} rd\,\phi ,
\]

\( \phi \in [0, 2\pi], \ r \in [0, \sqrt{b}] \).

Surface \( z = xy \)

\[
dS = \left( \begin{array}{c} y \\ x \\ -1 \end{array} \right) dx \, dy ,
\]

\[
dS = \sqrt{1 + r^2} rd\,\phi .
\]

Flat surface in \( xy \) plane

\[
dS = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) dx \, dy ,
\]

\[
dS = dx \, dy .
\]
Polar co-ordinates and general orthogonal curvilinear co-ordinates
52 $\nabla f$ in plane polar co-ordinates

$f(x(r, \phi), y(r, \phi))$ partially differentiable, then

$$\nabla f(x, y) = \frac{\partial f(x(r, \phi), y(r, \phi))}{\partial r} e_r + \frac{1}{r} \frac{\partial f(x(r, \phi), y(r, \phi))}{\partial \phi} e_\phi,$$

(126)

with $x = r \cos \phi, y = r \sin \phi$ and $e_r = \left(\begin{array}{c} \cos \phi \\ \sin \phi \end{array}\right), e_\phi = \left(\begin{array}{c} -\sin \phi \\ \cos \phi \end{array}\right)$. 
53 \( \nabla f \) in cylindrical polar coordinates

If \( f(x(r, \phi), y(r, \phi), z) \) partially differentiable, then

\[
\nabla f(x, y) = \frac{\partial f(x(r, \phi), y(r, \phi), z)}{\partial r} e_r + \frac{1}{r} \frac{\partial f(x(r, \phi), y(r, \phi), z)}{\partial \phi} e_\phi + \frac{\partial f(x(r, \phi), y(r, \phi), z)}{\partial z} e_z.
\]

with \( x = r \cos \phi \), \( y = r \sin \phi \) and 

\[
e_r = \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \quad \text{and} \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
54 General orthogonal curvilinear coordinates

Given a set of coordinates \( q_1, \ldots, q_n \) and let \( \mathbf{x} = (x_1(q_1, \ldots, q_n), \ldots, x_n(q_1, \ldots, q_n))^T \) denote the position vector. We construct basis vectors \( \mathbf{f}_j \) (depending on the position) in the following way:

\[
\mathbf{f}_j := \frac{1}{h_j} \frac{\partial \mathbf{x}}{\partial q_j} \quad \text{with} \quad h_j := \left\| \frac{\partial \mathbf{x}}{\partial q_j} \right\|, \tag{128}
\]

where the brackets around the indices in Eq. 128 denote that no summation convention should be applied. If the vectors \( \mathbf{f}_1, \ldots, \mathbf{f}_n \) are orthonormal then the co-ordinates \( q_1, \ldots, q_n \) are called orthogonal curvilinear co-ordinates. It is clear from Eq. 128 that the vectors \( \mathbf{f}_1, \ldots, \mathbf{f}_n \) are in general functions of the position in \( \mathbb{R}^n \).
\[ \nabla f(\mathbf{x}) = \sum_{i=1}^{n} \frac{1}{h_i} \frac{\partial f}{\partial q_i} \mathbf{f}_i \]

(129)
\[ \mathbf{e}_r = \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos \phi \cos \theta \\ \sin \phi \cos \theta \\ -\sin \theta \end{pmatrix}. \quad (131) \]

56 \( \nabla f \) spherical polar coordinates

\[ \nabla f(\mathbf{x}) = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta. \quad (132) \]
15  Vector operator identities

57 Differential vector operators\(^d\).

\[
\text{grad}f(\mathbf{x}) = \nabla f(\mathbf{x}) = e_i \frac{\partial f(\mathbf{x})}{\partial x_i}, \quad (133)
\]

\[
\text{div}\mathbf{F}(\mathbf{x}) = \nabla \cdot \mathbf{F}(\mathbf{x}) = \frac{\partial F_i(\mathbf{x})}{\partial x_i}, \quad (134)
\]

\[
\text{curl}\mathbf{F}(\mathbf{x}) = \nabla \times \mathbf{F}(\mathbf{x}) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k(\mathbf{x}) , \quad (135)
\]

\[
\nabla^2 f(\mathbf{x}) = \nabla \cdot \nabla f(\mathbf{x}) = \sum_i \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2} . \quad (136)
\]

\(^d\)With the exception of the curl these vector operators are all defined for any \(n \in \mathbb{N}\). The curl is only defined for \(n = 3\).
Vector operator identities

\[
\nabla(g(x)f(x)) = f \nabla g + g \nabla f, \quad (138)
\]

\[
\nabla \cdot (g(x)F(x)) = (\nabla g) F + g \nabla F, \quad (139)
\]

\[
\nabla \times (F(x) \times G(x)) = \nabla \cdot (G \nabla F + \nabla(G) F - (\nabla F)\nabla G - F \nabla G), \quad (140)
\]

\[
\nabla \times \nabla f(x) = 0. \quad (141)
\]
Vector operators in general orthogonal curvilinear co-ordinates in $\mathbb{R}^3$

The general orthogonal curvilinear co-ordinates in $\mathbb{R}^3$ are defined as in [54]. The differential vector operators are given by

\begin{equation}
\nabla f(\mathbf{x}) = \frac{1}{h_1} \frac{\partial f(\mathbf{x})}{\partial q_1} \mathbf{f}_1 + \frac{1}{h_2} \frac{\partial f(\mathbf{x})}{\partial q_2} \mathbf{f}_2 + \frac{1}{h_3} \frac{\partial f(\mathbf{x})}{\partial q_3} \mathbf{f}_3 ,
\end{equation}

\begin{equation}
\nabla \mathbf{F}(\mathbf{x}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (F_{q_1} h_2 h_3) + \frac{\partial}{\partial q_2} (h_1 F_{q_2} h_3) + \frac{\partial}{\partial q_3} (h_1 h_2 F_{q_3}) \right\} ,
\end{equation}

\begin{equation}
\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
    h_1 F_{q_1} & h_2 F_{q_2} & h_3 F_{q_3} \\
    \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\
    \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} 
\end{vmatrix} ,
\end{equation}

\begin{equation}
\nabla^2 f(\mathbf{x}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right\} ,
\end{equation}

where $F_{q_i}$ denotes the components of the vector field $\mathbf{F}$ with respect to the general orthogonal curvilinear basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. The formal determinant in Eq.144 has to be evaluated in such a way that the partial derivatives in the second row act on the functions in the third row but not on the factors in the first row.
Vector operators in plane polar co-ordinates

\[ \nabla f(x) = \frac{\partial f(x)}{\partial r} e_r + \frac{1}{r} \frac{\partial f(x)}{\partial \phi} e_\phi , \quad (146) \]

\[ \nabla \cdot \mathbf{F}(x) = \frac{1}{r} \frac{\partial F_r}{\partial r} + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} , \quad (147) \]

\[ \nabla^2 f(x) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} . \quad (148) \]
61 Vector operators in cylindrical polar co-ordinates

\[ \nabla f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f(\mathbf{x})}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f(\mathbf{x})}{\partial z} \mathbf{e}_z, \quad (149) \]

\[ \nabla \cdot \mathbf{F}(\mathbf{x}) = \frac{1}{\rho} \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}, \quad (150) \]

\[ \nabla \times \mathbf{F} = \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \mathbf{e}_\rho + \left( \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \mathbf{e}_\phi + \frac{1}{\rho} \left( \frac{\partial F_\phi}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \mathbf{e}_z, \quad (151) \]

\[ \nabla^2 f(\mathbf{x}) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \quad (152) \]
62 Vector operators in spherical polar co-ordinates

\[ \nabla f(x) = \frac{\partial f(x)}{\partial r} e_r + \frac{1}{r \sin \theta} \frac{\partial f(x)}{\partial \phi} e_\phi + \frac{1}{r} \frac{\partial f(x)}{\partial \theta} e_\theta, \quad (153) \]

\[ \nabla \cdot \mathbf{F}(x) = \frac{1}{r^2} \frac{\partial F_r}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial F_\theta}{\partial \theta}, \quad (154) \]

\[ \nabla \times \mathbf{F} = \frac{1}{r \sin \theta} \left( \frac{\partial \sin \theta F_\phi}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) e_r + \frac{1}{r} \left( \frac{\partial r F_\theta}{\partial r} - \frac{\partial F_\phi}{\partial \theta} \right) e_\phi + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial F_\phi}{\partial r} \right) e_\theta, \]

\[ \nabla^2 f(x) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right). \quad (155) \]
16 Green’s theorem in the plane

\[ \gamma_1(t) = \left( \frac{t}{f(x)} \right), \quad t \in [a, b], \quad (158) \]

\[ \gamma_2(t) = \left( \frac{b}{f(b) + t(g(b) - f(b))} \right), \quad t \in [0, 1], \quad (159) \]

\[ \gamma_3(t) = \left( \frac{t}{g(x)} \right), \quad t \in [a, b], \quad (160) \]

\[ \gamma_4(t) = \left( \frac{a}{f(a) + t(g(a) - f(a))} \right), \quad t \in [0, 1]. \quad (161) \]

This leads to

\[ \int_B \frac{\partial P(x,y)}{\partial y} \, d(x,y) = \int_a^b \int_{f(x)}^{g(x)} \frac{\partial P(x,y)}{\partial y} \, dy \, dx = \int_a^b P(x, g(x)) \, dx - \int_a^b P(x, f(x)) \, dx \]

\[ = \int_a^b P(\gamma_5(t)) \, dt - \int_a^b P(\gamma_1(t)) \, dt. \quad (162) \]

But since \( dx = dt \) on \( \gamma_1(t) \) and \( \gamma_5(t) \) and \( dx = 0 \) on \( \gamma_2(t) \) and \( \gamma_4(t) \) we find

\[ \int_B \frac{\partial P(x,y)}{\partial y} \, d(x,y) = - \int_{\gamma_1} P \, dx - \int_{\gamma_2} P \, dx - \int_{\gamma_3} P \, dx - \int_{\gamma_4} P \, dx \]

\[ = - \oint_{\partial B} P \, dx + 0 \, dy, \quad (163) \]

where \( \oint_{\partial B} P \, dx + 0 \, dy \) indicates a line integral along the closed boundary \( \partial B \) of \( B \) where the path has to be positively oriented (counterclockwise).
A similar calculation for a \textit{y-simple} surface \( C \) shows that

\[
\int_C \frac{\partial Q(x, y)}{\partial x} d(x, y) = \int_{\partial C} 0 \, dx + Qdy \quad . (164)
\]

In case we have a surface \( S \) which is both \textit{x-simple} as well as \textit{y-simple}; then we can put Eq. 163 and Eq. 164 together to obtain

\[
\int_S \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) d(x, y)
\]

\[
= \oint_{\partial S} P \, dx + Q \, dy = \int_{\partial S} \mathbf{F} \, d\mathbf{x} \quad . (165)
\]
Fig. xxxv  Splitting $S$ into $x$-simple and $y$-simple surfaces.
63 Green's theorem in the plane

The vector field \( \mathbf{F}(x, y) = (P(x, y), Q(x, y))^T \) is defined on the surface \( S \subset \mathbb{R}^2 \) with (piecewise) smooth boundary \( \partial S \), then

\[
\int_S \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) d(x, y) = \oint_{\partial S} \begin{pmatrix} P \\ Q \end{pmatrix} \, dx ,
\]

(166)

where \( \oint_{\partial S} \mathbf{F} \, dx \) denotes the line integral along the closed boundary \( \partial S \) of \( S \) using a positively oriented parameterisation (counterclockwise).
Example 5.1 Let

\[ I = \oint_C (x^2y \, dx + xy^2 \, dy), \]

for some curve \( C \) bounding a surface \( A \).

Green's theorem is

\[ \oint_C (P \, dx + Q \, dy) = \int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dS. \]

Setting

\[ P = x^2y \quad \text{and} \quad Q = xy^2. \]

\[ I = \int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy \]

\[ = \int_A (y^2 - x^2) \, dxdy. \]
17 Stokes’ theorem

64 Stokes’ theorem
The vector field $\mathbf{F} : G \subset \mathbb{R}^3 \to \mathbb{R}^3$ is defined on the domain $G$ such that the (piecewise) smooth surface $S$ is completely contained in $G$. We then find

$$
\int_S \nabla \times \mathbf{F} \cdot dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{x}.
$$

(168)

*There is of course an ambiguity in the sign of $dS$ as well as in the orientation of the closed path $\partial S$. If the surface $S$ is parametrised by $\Phi(u, v)$ then taking $dS = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \, du \, dv$ requires $\partial S$ to be parametrised in such a way that the underlying boundary of the parameter space in $(u, v)$ space is parametrised counter-clockwise. This will give the correct sign factors such that the identity Eq. 168 holds. In practice it is easier to calculate both sides of Eq. 168 using some parametrisations and decide afterwards by looking at the signs of the integrals whether we have used the correct orientations.*
Fig. xxi. Irrotational vector field $(x, y, 0)$.
Fig. xxxvii $(x, y, 0)$ projected into $xy$ plane.
Fig. xxxviii Vector field \((-y, x, 0)\) with curl \((0, 0, 2)\).
Fig. xxxix \((-y, x, 0)\) projected into \(xy\) plane.
Fig. xl Adding neighbouring whirlpool flows.
Fig. xli. Flow around the boundary $\partial S$. 
18 The divergence theorem

65 Divergence theorem

The vector field \( \mathbf{F} : G \subset \mathbb{R}^3 \to \mathbb{R}^3 \) is defined on the set \( G \) such that the volume \( \mathcal{V} \) with (piecewise) smooth boundary \( \partial \mathcal{V} \) is completely contained in \( G \). We then find

\[
\int_{\mathcal{V}} \nabla \mathbf{F} dV = \int_{\partial \mathcal{V}} \mathbf{F} \cdot d\mathbf{S},
\]

where the surface element \( d\mathbf{S} \) points in the direction of the outward normal to the closed boundary \( \partial \mathcal{V} \) of the volume \( \mathcal{V} \).
Fig. xlii  Solenoidal vector field (1, 1, 1).
Fig. xliii Solenoidal vector field \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\), for \(r \neq 0\).
Fig. xlv: Vector field \((x, |x|, |x|)\) with divergence 1 for \(x > 0\) and \(-1\) for \(x < 0\).
FIG. xlv Vector field \((x, y, z)\) with divergence 3.
VI LAPLACE’S EQUATION

19 Laplace’s equation
20 Green’s theorems

66 Green’s first theorem
The scalar fields \( \phi, \psi : G \subset \mathbb{R}^3 \to \mathbb{R} \) satisfy for any volume \( \mathcal{V} \subset G \)

\[
\int_{\mathcal{V}} \left( \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \right) \, dV = \oint_{\partial \mathcal{V}} \phi \nabla \psi \cdot dS
\]

\[
= \oint_{\partial \mathcal{V}} \phi \nabla \psi \cdot \mathbf{n} \, dS = \oint_{\partial \mathcal{V}} \frac{\partial \psi}{\partial \mathbf{n}} \, dS , \quad (186)
\]

where \( \mathbf{n} \) is the normalised outward normal vector to the surface \( \partial \mathcal{V} \) such that \( dS = \mathbf{n} dS \).
67 Green’s second theorem

The scalar fields $\phi, \psi : G \subset \mathbb{R}^3 \to \mathbb{R}$ satisfy for any volume $\mathcal{V} \subset G$

$$\int_\mathcal{V} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dV = \oint_{\partial \mathcal{V}} \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS,$$

where $n$ is the normalised outward normal vector to the surface $\partial \mathcal{V}$ such that $dS = ndS$.\n
21 Harmonic functions

A function $\phi$, is called harmonic on the volume $\mathcal{V}$ if it satisfies Laplace’s equation on $\mathcal{V}$:
$$\nabla^2 \phi(x) = 0 \quad \forall x \in \mathcal{V}.$$
Harmonic functions with trivial boundary conditions

The function $\psi$ is harmonic on the volume $\mathcal{V} \subset \mathbb{R}^3$. If $\psi(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \partial \mathcal{V}$ then

$$\psi(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathcal{V}. \quad (190)$$
Uniqueness theorem for harmonic functions with Dirichlet boundary conditions

The functions $\phi_1$ and $\phi_2$ are harmonic on the volume $\mathcal{V}$ and satisfy the same Dirichlet boundary conditions $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x} \in \partial \mathcal{V}$ then

$$
\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{V}.
$$

(191)
71 Constant harmonic functions
The function \( \psi \) is harmonic on the volume \( \mathcal{V} \). If \( \psi(x) = c \ \forall x \in \partial \mathcal{V}, \ c \in \mathbb{R} \) then

\[
\psi(x) = c \ \forall x \in \mathcal{V}.
\] (192)
72 Boundary conditions
A real valued function $\psi$ defined on the volume $\mathcal{V}$ is said to satisfy

(i) Dirichlet boundary conditions if $\psi(x)$ is given as a function on the boundary $\partial \mathcal{V}$;
(ii) von Neumann boundary conditions if the directional derivative $\frac{\partial \psi}{\partial n}$ is given on $\partial \mathcal{V}$;
(iii) boundary conditions of the mixed type if the combination $\frac{\partial \psi(x)}{\partial n} + f(x)\psi(x)$ is given on the $\partial \mathcal{V}$.

*John von Neumann, 1903-1957.*
73 Uniqueness theorem for von Neumann boundary conditions

The functions $\phi_1$ and $\phi_2$ are harmonic on the volume $\mathcal{V}$ and satisfy the same von Neumann boundary conditions $\frac{\partial \phi_1(x)}{\partial n} = \frac{\partial \phi_2(x)}{\partial n}$ for all $x \in \partial \mathcal{V}$ then

$$
\phi_1(x) = \phi_2(x) + c \quad \forall x \in \mathcal{V},
$$

for some constant $c \in \mathbb{R}$. 
74 Harmonic average

A harmonic function \( \psi \) is harmonic on the volume \( \mathcal{V} \) then it satisfies

\[
\iiint_{\mathcal{V}} \nabla \psi \cdot d\mathbf{S} = 0.
\]  

(194)
75 Mean value theorem for harmonic functions

A harmonic function $\psi$ on $V \subset \mathbb{R}^3$ satisfies

$$\psi(c) = \frac{1}{4\pi R^2} \int_{S_R(c)} \psi(x) dS ,$$

(195)

where $S_R(c)$ is the surface of a sphere with radius $R$ centred at the point $c$. 
Global extrema of harmonic functions
The global maximum and the global minimum of a harmonic function $\psi$ defined on the volume $\mathcal{V}$ lies on the boundary $\partial\mathcal{V}$ unless $\psi$ is constant on the whole of $\mathcal{V}$. 
Harmonic functions with the same Dirichlet boundary conditions

Among all (differentiable) functions $\mathcal{F}_f$ on $\mathcal{V}$ with the same Dirichlet boundary conditions $f(x)$ on $\partial \mathcal{V}$ as the harmonic function $\psi$ we find that $\psi$ minimises the integral over the norm of the gradient:

$$\int_\mathcal{V} \| \nabla \omega \|^2 dV \geq \int_\mathcal{V} \| \nabla \psi \|^2 dV,$$

for all functions $\omega \in \mathcal{F}_f$. 

22 Gauss’ flux theorem and Gauss’ law

78 Gauss’ flux theorem
The function \( \phi \) is a solution to Poisson’s equation \( \nabla^2 \phi(\mathbf{x}) = f(\mathbf{x}) \) on the domain \( G \) if and only if the vector field \( \mathbf{F} := \nabla \phi \) satisfies

\[
\int_{\partial \mathcal{V}} \mathbf{F} \cdot d\mathbf{S} = \int_{\mathcal{V}} f(\mathbf{x}) dV,
\]

for any (piecewise smooth) volume \( \mathcal{V} \subset G \).
**79 Gauss' law**

*For any volume \( \mathcal{V} \) and any \( \mathbf{x}_0 \in \mathbb{R}^3 \) with \( \mathbf{x}_0 \notin \partial \mathcal{V} \)*

\[
\int_{\partial \mathcal{V}} \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3} \, d\mathbf{s} = \begin{cases} 
4\pi & , \mathbf{x}_0 \in \mathcal{V} \\
0 & , \mathbf{x}_0 \notin \mathcal{V} 
\end{cases}
\]  \hspace{1cm} (203)
23 Poisson’s equation

80 Solutions to Poisson’s equation

For Poisson’s equation $\nabla^2 \phi = f(\mathbf{x})$ defined on some finite volume $\mathcal{V}$ we can set $f(\mathbf{x}) = 0$ outside the volume $\mathcal{V}$ and obtain a solution\(^b\)

$$
\phi(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{\| \mathbf{x} - \mathbf{y} \|} d(y_1, y_2, y_3),
$$

(207)

where the integral is taken over the whole $\mathbb{R}^3$. This solution satisfies the boundary conditions $\phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. The corresponding vector field is given by

$$
\mathbf{F}(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})(\mathbf{x} - \mathbf{y})}{\| \mathbf{x} - \mathbf{y} \|^3} d(y_1, y_2, y_3).
$$

(208)

\(^b\)Note that $\int_{\mathbb{R}^3}$ is in fact only an integral $\int_{\mathcal{V}}$.  

VII \textbf{Cartesian Tensors in } \mathbb{R}^3

\textbf{Example 7.1} Consider the two bases (which happen to be orthonormal bases) of \( \mathbb{R}^3 \):

\[
B = \begin{Bmatrix}
\left( \frac{1}{2} \sqrt{2} \right), \\
\left( -\frac{1}{2} \sqrt{2} \right), \\
\left( 0 \right)
\end{Bmatrix},
\]

\[
(219)
\]

\[
B' = \begin{Bmatrix}
\left( \frac{1}{2} \sqrt{2} \right), \\
\left( 0 \right), \\
\left( 0 \right)
\end{Bmatrix},
\]

\[
(220)
\]

We can write the vectors of \( B \) in terms of \( B' \):

\[
\begin{align*}
\left( \frac{1}{2} \sqrt{2} \right) &= \frac{1}{2} \left( \frac{1}{2} \sqrt{2} \right) - \frac{1}{2} \sqrt{2} \left( 0 \right) - \frac{1}{2} \left( -\frac{1}{2} \sqrt{2} \right), \\
\left( -\frac{1}{2} \sqrt{2} \right) &= \frac{1}{2} \left( -\frac{1}{2} \sqrt{2} \right) + \frac{1}{2} \sqrt{2} \left( 1 \right) - \frac{1}{2} \left( -\frac{1}{2} \sqrt{2} \right), \\
\left( 0 \right) &= \frac{1}{2} \sqrt{2} \left( 0 \right) + \left( 0 \right) + \frac{1}{2} \sqrt{2} \left( -\frac{1}{2} \sqrt{2} \right).
\end{align*}
\]

We now put these components into the columns of the matrix \( L \):

\[
L = \begin{pmatrix}
-\frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \sqrt{2}
\end{pmatrix}.
\]

\[
(221)
\]
24 Cartesian Tensors

81 Vectors

A vector is an object represented by components $x_i$ which transform according to

$$x'_i = L_{ij} x_j,$$  \hspace{1cm} (233)

under the orthogonal basis transformation described by $L$. Eq. 233 can be inverted to

$$x_i = L_{ji} x'_j.$$  \hspace{1cm} (234)
Matrices

A matrix is an object represented by components $x_{ij}$ which transform according to

$$x'_{ij} = L_{ik}L_{jl}x_{kl}, \tag{235}$$

under the orthogonal basis transformation described by $L$. Eq. 235 can be inverted to

$$x_{ij} = L_{kl}L_{lj}x'_{kl}. \tag{236}$$
83 Scalars

A scalar is an object represented by a real number $x$ which stays the same under the orthogonal basis transformation described by $L$. 


84 Cartesian tensors

A Cartesian tensor of rank $^b k \in \mathbb{N}_0$ is an object represented by the components $x_{i_1 \ldots i_k}$ which transform according to

$$x'_{i_1 i_2 \ldots i_k} = L_{i_1 j_1} L_{i_2 j_2} \cdots L_{i_k j_k} x_{j_1 j_2 \ldots j_k},$$

(237)

under the orthogonal basis transformation described by $L$. In particular, a scalar is a rank 0 tensor, a vector is a rank 1 tensor and a matrix is a rank 2 tensor. Eq. 237 can be inverted to

$$x_{i_1 i_2 \ldots i_k} = L_{j_1 i_1} L_{j_2 i_2} \cdots L_{j_k i_k} x'_{j_1 j_2 \ldots j_k}.$$  

(238)

---

$^a$ We call these tensors Cartesian tensors since we assumed that the basis transformation $L$ is orthogonal with $\det L = 1$. Without this simplification we would find for general non-orthogonal $L$ two different types of indices: indices transforming with $L$ and others transforming with $L^{-1}$. The former type of index is called covariant and the latter is called contravariant. A matrix is therefore in this general setting a once covariant, once contravariant tensor whilst a vector is simply a once covariant tensor. In order to include in the notation which index is covariant and which index is contravariant we would then put the covariant indices down and the contravariant indices up. Hence, a vector would still be written as $x_i$ whilst a matrix would be written as $a^i_i$. For our applications Cartesian basis transformations are sufficient and therefore we do not need to distinguish these different types of indices.

$^b$ The rank of a tensor must not be confused with the rank of a matrix.
Example 7.2 The dot product $s = x_i y_i$ of two rank 1 tensors (vectors) $x_i$ and $y_i$ is a rank 0 tensor (a scalar). This can easily be seen by taking the dot product $s' = x'_i y'_i$ of the components with respect to $B'$ and insert the transformation properties $x'_i = L_{ij} x_j$ and $y'_i = L_{ik} y_k$: $x'_i y'_i = L_{ij} x_j L_{ik} y_k = x_j y_k = x_j y_j$ and therefore $s' = s$. 
Example 7.3 An object which is represented by the components of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(239)

in the basis $B$ and by the matrix

$$A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(240)

in the basis $B'$ cannot be a tensor. This can easily be seen since the determinant $\det A = 2$ but the determinant $\det A = 0$. We know from Algebra and Geometry that the determinant of a matrix stays the same under a basis transformation, therefore there cannot be a basis transformation of the type of Eq. 235 (not even of the type of Eq. ??) connecting these two matrices.

In the same way the matrix

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(241)

in the basis $B$ and the matrix

$$C' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(242)

in the basis $B'$ cannot represent a tensor since the trace $\text{tr}(C) = 3$ and the trace $\text{tr}(C') = -1$ and we again know from Algebra and Geometry that a basis transformation leaves the trace invariant. So, $C$ and $C'$ cannot be connected by a basis transformation.

Another way of spotting that $A$ and $A'$ cannot represent a tensor is since the rank (this time the matrix rank, not the tensor rank!) of the matrix $A$ is 3 whilst the rank of the matrix $A'$ is 2 and again, the rank of a matrix has to be the same under basis transformation which shows that there is no basis transformation connecting $A$ and $A'$. The same argument would obviously not work for $C$ and $C'$ since they both have (matrix) rank 3 but still do not represent a tensor as can be seen from the different traces.
Example 7.4 A physical entity represented in every Cartesian frame by the array of numbers \(\delta_{ij}\) is a rank 2 tensor. In frame 1 the entity is represented by \(\delta_{ij}\). We then have to check that \(L_{ik}L_{jk}\delta_{kl}\) equals the new components (in frame 2) which by definition are again the same \(\delta_{ij}\). But this is obvious from Eq. ??: \(L_{ik}L_{jk}\delta_{kl} = L_{ik}L_{jk} = \delta_{ij}\) and hence it is a rank 2 tensor. This tensor has the property that the components are exactly the same in each Cartesian frame. A tensor with this property is called isotropic. We will study isotropic tensors in Sec. 28.

A physical entity represented in every Cartesian frame by the components \(\epsilon_{ijk}\) is a rank 3 tensor. In frame 1 the entity is represented by \(\epsilon_{ijk}\). We then have to check that \(L_{il}L_{jm}L_{kn}\epsilon_{mn}\) equals the new components (in frame 2) which by definition are again the same \(\epsilon_{ijk}\). If we let \(L_i\) represent the \(i\)-th row of the basis transformation matrix \(L\), then \(L_{il}L_{jm}L_{kn}\epsilon_{mn} = [l_i, l_j, l_k]\) with \([l_i, l_j, l_k] = l_i(l_j \times l_k)\) being the scalar triple product. If any of the indices \(i, j, k\) are the same, then obviously \([l_i, l_j, l_k] = 0\) otherwise exactly one of them is equal to 1, one equal to 2 and one equal to 3. \([l_1, l_2, l_3] = \det L = 1\) since \(L\) is a rotation and all the other such cases would be +1 if they can be reached using an even permutation of the vectors \(l_1, l_2, l_3\) and −1 if we require an odd permutation. Taking all this together we find that \([l_i, l_j, l_k]\) reflects exactly the definition of \(\epsilon_{ijk}\) and hence \([l_i, l_j, l_k] = \epsilon_{ijk}\). But this just means \(L_{il}L_{jm}L_{kn}\epsilon_{mn} = \epsilon_{ijk}\) and therefore \(\epsilon_{ijk}\) is a rank 3 tensor (which is like \(\delta_{ij}\) isotropic).
Example 7.5 A physical entity is described in frame 1 by the matrix

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{243} \]

in and in frame 2 by the matrix

\[ A' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \alpha \end{pmatrix}, \tag{244} \]

with some \( \alpha \in \mathbb{R} \). Frame 2 is obtained from frame 1 by rotating frame 1 around the z-axis by \( \frac{\pi}{2} \) in positive direction (using the right hand rule). This entity could be a tensor. \( L^{-1} \) contains in its columns the frame 2 basis vectors written in terms the basis of frame 1. Therefore the first frame 2 basis vector has the components \((0, 1, 0)^T\) in frame 1, the second frame 2 basis vector has the components \((-1, 0, 0)^T\) in frame 1 and finally the third frame 2 basis vector has the components \((0, 0, 1)^T\) in frame 3. Therefore

\[ L^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{245} \]

and

\[ L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{246} \]

We can easily check that \( A' = LAL^{-1} \) if and only if \( \alpha = -1 \) and therefore \( A \) could represent a rank 2 tensor (a matrix) in the case \( \alpha = -1 \). But we need to keep in mind that we have of course only checked one particular rotation rather than all rotations (in other words just one particular basis transformation \( L \) rather than all basis transformations \( L \)). Therefore we can only say that \( A \) 'could represent' a tensor rather than 'it is' a tensor.

If we assume that the matrix \( A \) does represent a tensor and frame 3 is obtained by rotating frame 1 around the x-axis by \( +\frac{\pi}{2} \). What is the matrix \( A'' \) representing the tensor in frame 3? The basis transformation \( \hat{L} \) is obviously given by

\[ \hat{L}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{247} \]

and

\[ \hat{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{248} \]

For example the second axis of frame 3 is in the yz-plane of frame 1 at an angle of \( \frac{\pi}{4} \) to the x-axis and therefore has components \((0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T\) in frame 1 which defines the second column of \( \hat{L}^{-1} \). Since \( A \) represents a tensor we need \( A'' = \hat{L}AL^{-1} \) and hence

\[ A'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}. \tag{249} \]
Example 7.6 The partial derivatives $\frac{\partial}{\partial x_i}$ represent a rank 1 tensor (i.e., a vector). We use the chain rule $\frac{\partial}{\partial x_i} = \frac{\partial x_j}{\partial x_i} \frac{\partial}{\partial x_j}$. Eq. 234 $x_j = L_{ij}x_i'$ leads to $\frac{\partial x_j}{\partial x_i} = L_{ij}$ and therefore $\frac{\partial}{\partial x_i} = L_{ij} \frac{\partial}{\partial x_j}$ which shows that $\frac{\partial}{\partial x_i}$ and therefore $\nabla$ is a rank 1 tensor.
Example 7.7 The partial derivatives $\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$ represent a rank 3 tensor. This time we use the chain rule three times to convert the third order derivatives in the new frame to third order derivatives in the old frame: $\frac{\partial^3}{\partial x_i' \partial x_j' \partial x_k'} = \frac{\partial x_i}{\partial x_i'} \frac{\partial x_j}{\partial x_j'} \frac{\partial x_k}{\partial x_k'} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$. Eq. 234 $x_i = L_{il} x_i'$ leads to $\frac{\partial x_i}{\partial x_i'} = L_{il}$ and similarly $\frac{\partial x_m}{\partial x_j} = L_{jm}$ and $\frac{\partial x_n}{\partial x_k} = L_{kn}$, and therefore $\frac{\partial^3}{\partial x_i' \partial x_j' \partial x_k'} = L_{il} L_{jm} L_{kn} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$ which shows that $\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$ is a rank 3 tensor. Note that Schwarz’s theorem [1] says that higher order partial derivatives are independent of the order of differentiation. Therefore this rank 3 tensor stays the same under exchange of any of the indices. Such a tensor is called totally symmetric and we will study totally symmetric tensors in Sec. 29.
Example 7.8 The differential $dx_i$ is a rank 1 tensor, a vector. This is clear from the chain rule $dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j = L_{ij} dx_j$ which follows from $x'_i = L_{ij} x_j$. 
Example 7.9 Let $\mathcal{V}$ describe a volume in $\mathbb{R}^3$. The volume integral $\int_{\mathcal{V}} dV$ is a rank 0 tensor (a scalar) since we find in frame $2$ the volume integral $\int_{\mathcal{V}} dV' = \int_{\mathcal{V}} d(x_1', x_2', x_3') = \int_{\mathcal{V}} \frac{dx'(1, x_2', x_3')}{dx(1, x_2, x_3)} d(x_1, x_2, x_3)$ with the Jacobian $\frac{dx'(1, x_2', x_3')}{dx(1, x_2, x_3)} = |\det L|$ using $x_i = L_{ij}x_j$. But $\det L = 1$ for the rotation $L$ and therefore $\int_{\mathcal{V}} dV' = \int_{\mathcal{V}} dV$.

In the same way we see that the integral $\int_{\mathcal{V}} x_i dV$ is a rank 1 tensor (a vector): $\int_{\mathcal{V}} x_i' dV' = \int_{\mathcal{V}} x_i d(x_1', x_2', x_3') = \int_{\mathcal{V}} L_{ij}x_j \frac{dx'(1, x_2', x_3')}{dx(1, x_2, x_3)} d(x_1, x_2, x_3) = L_{ij} \int_{\mathcal{V}} x_j dV$. 
25 Tensor Rules and Properties

85 Linear combinations of tensors
The linear combination \( \alpha x_{i_1i_2...i_k} + \beta y_{i_1i_2...i_k} \) (\( \alpha, \beta \in \mathbb{R} \)) of the two rank \( k \) tensors \( x_{i_1i_2...i_k} \) and \( y_{i_1i_2...i_k} \) is again a tensor of rank \( k \).
Uniqueness theorem
If two rank $k$ tensors $x_{i_1i_2...i_k}$ and $y_{i_1i_2...i_k}$ are equal in one frame: $x_{i_1i_2...i_k} = y_{i_1i_2...i_k}$ then they are equal in all frames.
87 Outer product

The outer product $x_{i_1i_2...i_k}y_{j_1j_2...j_l}$ of the tensors $x_{i_1i_2...i_k}$ of rank $k$ and $y_{j_1j_2...j_l}$ of rank $l$ is a tensor of rank $k+l$. 
Example 7.10 If we take the vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R}^3$ with the components $v_i$ and $u_j$ respectively then the product $v_iu_j$ define the components of a matrix. This is easily seen by looking at the transformation properties of $v_iu_j$: $v'_iu'_j = L_{ik}v_kL_{jl}u_l = L_{ik}L_{jl}v_ku_l$ which matches the transformation properties Eq. 235 of a matrix under the rotation $L$.

In the case of vectors and matrices we can of course re-write the transformation properties in terms of matrix multiplication. The matrix product $\mathbf{vu}^T = A$ defines a 3 by 3 matrix $A$. Under the rotation $L$ we obtain $\mathbf{v}' = L\mathbf{v}$ and $\mathbf{u}' = L\mathbf{u}$ and therefore $A' = \mathbf{v}'\mathbf{u'}^T = (L\mathbf{v})(L\mathbf{u})^T = L\mathbf{vu}^TL^T = LAL^{-1}$ noting that $L^T = L^{-1}$. Hence $A' = LAL^{-1}$ which defines $A$ as a matrix.

---

*Note that $\mathbf{vu}^T$ is a 3 by 3 matrix whilst $\mathbf{v}^\top\mathbf{u}$ would simply be a number, the dot product $\mathbf{v}\cdot\mathbf{u}$. 
Inner product

The inner product of the tensors \( x_{i_1 i_2 \ldots i_k} \) of rank \( k \) and \( y_{j_1 j_2 \ldots j_l} \) of rank \( l \) is the product of these components with at least one index being equal and summed over, e.g. \( x_{ii_2 \ldots i_k}y_{ij_2 \ldots j_l} \). In the case that in the inner product \( m \) pairs of indices are the same and are being summed over then the inner product is a tensor of rank \( k + l - 2m \).
Example 7.11 The product of the two matrices $A$ and $B$ is described by the inner product $a_{ij} b_{jk}$ of the rank 2 tensors $a_{ij}$ and $b_{mn}$. The inner product $a_{ij} b_{jk}$ with the second index of $a_{ij}$ and the first index of $b_{mn}$ being set equal and summed over is therefore a rank $2 + 2 - 2 = 2$ tensor, i.e. a matrix. In matrix notation this is of course obvious from $A'B' = (LAL^{-1})(LBL^{-1}) = LAL^{-1}LB^{-1}L^{-1} = LABL^{-1}$.

$a_{ij} b_{kj}$ is another inner product that can be constructed using the tensors $A$ and $B$. In matrix notation this can be written as $AB^T$ since $(AB^T)_{ik} = (A)_{ij}(B^T)_{jk} = a_{ij} b_{kj}$. In matrix notation the tensor property of $AB^T$ is shown by $A'B'' = (LAL^{-1})(LBL^{-1})^T = LAL^{-1}(L^{-1})^T B^T L^T = LAL^{-1}LB^T L^{-1} = LAB^T L^{-1}$ since $L^T = L^{-1}$ for the rotation $L$.

$a_{ij} b_{ji}$ is another inner product constructed from the tensors $A$ and $B$. This is in fact the trace $tr(AB)$ of the matrix product $AB$. This time the inner product has two pairs of indices set equal and summed over them and we obtain a tensor of rank $2 + 2 - 2 	imes 2 = 0$, hence a scalar. Noting that within the trace matrices can be commuted $tr(AB) = tr(BA)$ we can easily see in matrix notation $tr(A'B') = tr(LAL^{-1} LBL^{-1}) = tr(AL^{-1} LBL^{-1} L) = tr(AB)$ and therefore $tr(AB)$ is indeed a scalar tensor.
89 Contraction

If we set two of the indices (the $m$-th and $n$-th index) of the tensor $x_{i_1i_2...i_k}$ of rank $k$ equal and sum over this common index (i.e. $i_m = i_n = i$) then we call it the contraction of the tensor $x_{i_1i_2...i_k}$ with respect to the indices $m$ and $n$ ($m \neq n$). The contraction is a tensor of rank $k - 2$. For example, $x_{i_3i_3...i_k}$ is the contraction of $x_{i_1i_2...i_k}$ with respect to the first and second index.
**Example 7.12** The trace \( \text{tr} (A) \) of the matrix \( A \) is a rank 0 tensor, a scalar, and is obviously the contraction of the rank 2 tensor \( a_{ij} \): \( \text{tr} (A) = a_{ii} \).

The contraction of the rank 2 tensor \( \frac{\partial^2}{\partial x_0 \partial x_j} \) is \( \frac{\partial^2}{\partial x_i \partial x_i} = \nabla^2 \) and therefore the Laplace operator \( \nabla^2 \) is a rank 0 tensor, a scalar.
Example 7.13 The matrix product $AB$ is the inner product of $a_{ij}$ and $b_{mn}$ with the second index of $A$ and the first index of $B$ being set equal: $a_{ij}b_{kl}$. But this is obviously the same as the contraction of the outer product $a_{ij}b_{kl}$ with respect to the indices $j$ and $k$ and is also the same as the inner product of the rank 4 tensor $a_{ij}b_{kl}$ with the rank 2 tensor $\delta_{mn}$, setting the pairs $i$, $m$ and $j$, $n$ equal.

The double contraction of the rank 6 tensor $\varepsilon_{ijk}\varepsilon_{jkl}$ is a rank $6 - 4 = 2$ tensor. This contraction is in fact equal to the rank 2 tensor $2\delta_{kl}$: $\varepsilon_{ijk}\varepsilon_{jkl} = \delta_{jj}\delta_{kl} - \delta_{j}\delta_{kj} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$.
The Quotient Theorem

Quotient theorem for tensors

Given an array of numbers $a_{i_1i_2...i_k}$. If $a_{i_1i_2...i_m-1i_mi_m+1...i_n}x_{j_1j_2...j_m-1j_mj_m+1...j_k}$ (where we have contracted over the pair of indices $i_m$ and $j_n$ for some $m$, $n$) is a tensor of rank $k + l - 2$ for all rank $l$ tensors $x_{i_1i_2...i_l}$ then $a_{i_1i_2...i_k}$ is a rank $k$ tensor itself.
Example 7.14 Given an arbitrary vector $v$ then $\delta_{ij}v_j = v_i$ and therefore $\delta_{ij}v_j$ is a vector. This is another proof for $\delta_{ij}$ being a rank 2 tensor.

The outer product of the differentials $dx_idx_j$ is a rank 2 tensor using example 7.8 and the definition of the outer product. If we assume that the array of numbers $g_{ij}$ leads to a scalar when double contracted with the arbitrary differentials $dx_idx_j$: $g_{ij}dx_idx_j$ is a scalar, then $g_{ij}$ is a rank 2 tensor, called the metric tensor. This scalar tensor is denoted by $ds^2 = g_{ij}dx_idx_j$. We will study the metric tensor in Sec. 30.
Rank 2 Tensors in $\mathbb{R}^3$ - Examples and Properties
91 Conductivity tensor and Ohm's \textsuperscript{d} law

The conductivity tensor $\sigma$ is a rank 2 tensor describing the electric current $j$ generated by the electric field $E$ in a conducting material: $j = \sigma E$ and in suffix notation

$$j_m = \sigma_{mn} E_n.$$  \hspace{1cm} (254)

If the conductor is inhomogeneous then $\sigma$ is a function of the position on the conductor. $j = \sigma E$ is called Ohm's law.

In case the conductor is isotropic (i.e. it conducts in all directions equally well) then $j$ will obviously always be parallel to $E$ with the same proportionally factor and therefore $\sigma_{ij} = \zeta \delta_{ij}$ for an isotropic material (c.f. example 7.18).

\textsuperscript{d}Georg Ohm, 1789-1854.
92 Centre of mass

The mass $M = \int_\mathcal{V} \rho(\mathbf{x})dV$ of the massive object described by the volume $\mathcal{V}$ and mass density $\rho(\mathbf{x})$ is a rank 0 tensor. Its centre of mass $c_i = \frac{1}{M} \int_\mathcal{V} \rho(\mathbf{x})x_idV$ is a rank 1 tensor, a vector $\mathbf{c}$. 
93 Inertia Tensor

The rank 2 tensor

\[ \Theta_{ij}(0) = \int_V \rho(\mathbf{x}) (x_k x_k \delta_{ij} - x_i x_j) dV \]  \hspace{1cm} (259)

is called the inertia tensor about the origin. It relates the angular momentum \( \mathbf{L} \) of a rigid body about the origin with the angular velocity \( \omega \):

\[ \mathbf{L} = \Theta(0) \cdot \omega, \]  \hspace{1cm} (260)

or in index notation

\[ L_i = \Theta_{ij}(0) \omega_j. \]  \hspace{1cm} (261)
Example 7.15 Let us take the homogeneous massive cylinder $x^2 + y^2 \leq 1$ and $-1 \leq z \leq 1$ with mass density $\rho(x) = 1$. The mass of the cylinder is $M = 2\pi$ which is equal to its volume. We write the inertia tensor in cylindrical polar coordinates:

$$
\Theta_{ij}(0) = \int_V \left( (r^2 + z^2)\delta_{ij} - x_i x_j \right) dV .
$$

The off-diagonal components involve an integral over either $\sin \phi \cos \phi$ or just $\cos \phi$ or $\sin \phi$. All these integrals over a whole period of $2\pi$ vanish and therefore all off-diagonal components of $\Theta_{ij}(0)$ vanish.

$$
\Theta_{11}(0) = \int_0^1 \int_{-1}^1 \int_0^{2\pi} \left( (r^2 + z^2) - r^2 \cos^2 \phi \right) r d\phi dz dr
$$

$$
= \int_0^1 \left[ 2\pi(2r^3 + \frac{2}{3}r) - \pi r^3 \right] dr = \frac{7}{6} \pi = \Theta_{22}(0) ,
$$

$$
\Theta_{33}(0) = \int_0^1 \int_{-1}^1 \int_0^{2\pi} \left( (r^2 + z^2) - z^2 \right) r d\phi dz dr = \int_0^1 [2\pi r^3] dr = \pi .
$$

We therefore find the inertia tensor

$$
\Theta(0) = \begin{pmatrix}
\frac{7}{6}\pi & 0 & 0 \\
0 & \frac{7}{4}\pi & 0 \\
0 & 0 & \pi
\end{pmatrix} .
$$

(262)

The principal axes are the $z$-axis and any direction in the $xy$-plane.
**Parallel axis theorem**

The inertia tensors about the origin \( \mathbf{0} \) and about the centre of mass \( \mathbf{c} \) are related by

\[
\Theta_{ij}(\mathbf{0}) = \Theta_{ij}(\mathbf{c}) + M(\delta_{ij} \| \mathbf{c} \|^2 - c_i c_j). \tag{267}
\]

The inertia tensors about the point \( \mathbf{b} \) and about the centre of mass \( \mathbf{c} \) are therefore related by

\[
\Theta_{ij}(\mathbf{b}) = \Theta_{ij}(\mathbf{c}) + M \left( \delta_{ij} \| \mathbf{b} - \mathbf{c} \|^2 - (b_i - c_i)(b_j - c_j) \right). \tag{268}
\]
Example 7.16 We use the parallel axis theorem to find the inertia tensor about $\mathbf{b} = (1, 0, 0)$. The centre of mass of the cylinder is obviously $\mathbf{c} = (0, 0, 0)$. This leads to

$$M \left( \delta_{ij} \| \mathbf{b} - \mathbf{c} \|^2 - (b_i - c_i)(b_j - c_j) \right) = M \left( \delta_{ij} - \delta_{i1} \delta_{j1} \right)$$

$$\Theta((1, 0, 0)) = \begin{pmatrix} \frac{7}{6} \pi & 0 & 0 \\ 0 & \frac{2}{6} \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 \pi & 0 \\ 0 & 0 & 2 \pi \end{pmatrix} = \begin{pmatrix} \frac{7}{6} \pi & 0 & 0 \\ 0 & \frac{12}{6} \pi & 0 \\ 0 & 0 & 3 \pi \end{pmatrix}.$$ 

Therefore the principal axes are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (note that this is for the rotation about the point $(1, 0, 0)$).

We now want to find the inertia tensor about $\mathbf{b} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$:

$$M \left( \delta_{ij} \| \mathbf{b} - \mathbf{c} \|^2 - (b_i - c_i)(b_j - c_j) \right) = M \left( \delta_{ij} - \frac{1}{2} (\delta_{i1} + \delta_{22})(\delta_{j1} + \delta_{j2}) \right)$$

$$\Theta((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)) = \begin{pmatrix} \frac{7}{6} \pi & 0 & 0 \\ 0 & \frac{7}{6} \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} + \begin{pmatrix} \pi & -\pi & 0 \\ -\pi & \pi & 0 \\ 0 & 0 & 2 \pi \end{pmatrix} = \begin{pmatrix} \frac{13}{6} \pi & -\pi & 0 \\ -\pi & \frac{13}{6} \pi & 0 \\ 0 & 0 & 3 \pi \end{pmatrix}.$$ 

Therefore the principal axes are $(0, 0, 1)$, $(1, 1, 0)$ and $(1, -1, 0)$. 
Isotropic Tensors

A rank $k$ tensor $x_{i_1i_2...i_k}$ is called isotropic if $x'_{i_1i_2...i_k} = x_{i_1i_2...i_k}$ for all rotations $L$. In other words, an isotropic tensor has exactly the same components in all Cartesian frames.
Rank 0 and rank 1 isotropic tensors

Rank 0 tensors (scalars) are by definition always isotropic but there is only one isotropic rank 1 tensor (a vector) which is the 0 vector.
Example 7.17 In example 7.9 we showed that $\int_V x_i dV$ is a rank 1 tensor. If $V$ is a sphere centered at the origin then due to the symmetry of the volume it is clear that the integral has to be isotropic: $x_i$ is obviously just a dummy variable for the integration which under a rotation due to the spherical symmetry of $V$ always contributes in the same way to the integral. Therefore $\int_V x_i dV$ is an isotropic rank 1 tensor but since 0 is the only isotropic rank 1 tensor we immediately find $\int_V x_i dV = 0$. 
\[
\begin{pmatrix}
    a_{22} & -a_{21} & a_{23} \\
    -a_{12} & a_{11} & -a_{13} \\
    a_{32} & -a_{31} & a_{33}
\end{pmatrix}
= \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix}.
\]
97 Rank 2 isotropic tensors
The only rank 2 isotropic tensors (in 3 dimensions) are given by $\lambda \delta_{ij}$ for $\lambda \in \mathbb{R}$.

*In two dimensions the most general rank 2 isotropic tensor can be written as $\lambda \delta_{ij} + \mu \epsilon_{ij}$, where $\epsilon_{ij}$ is a rank 2 isotropic tensor with $\epsilon_{12} = 1$, $\epsilon_{21} = -1$ and $\epsilon_{11} = \epsilon_{22} = 0$ (i.e. antisymmetric).*
Example 7.18 The conductivity tensor of an isotropic material (a conductor conducting in all directions equally well) is an isotropic rank 2 tensor. Therefore $\sigma$ is proportional to $\delta_{ij}$ for an isotropic conductor.
Example 7.19 If $\mathcal{V}$ is a sphere of radius $R$ centred at the origin then $\int_{\mathcal{V}} x_i x_j dV$ is obviously an isotropic rank 2 tensor (the fact that it is a rank 2 tensor can be shown as in example 7.9; the fact that it is isotropic follows from the spherical symmetry of $\mathcal{V}$). Using [97] we find that $\int_{\mathcal{V}} x_i x_j dV = \lambda \delta_{ij}$ for some $\lambda \in \mathbb{R}$. We can compute $\lambda$ by contracting the tensor:

$$\lambda \delta_{ii} = 3\lambda = \int_{\mathcal{V}} x_i x_i dV = \int_{\mathcal{V}} r^2 dV = \int_0^R \int_0^{2\pi} \int_0^\pi r^4 \sin \theta d\theta d\phi dr = \frac{4\pi}{5} R^5.$$

Therefore, we find

$$\int_{\mathcal{V}} x_i x_j dV = \frac{4\pi}{15} R^5 \delta_{ij}.$$
98 Rank 3 isotropic tensors

The only rank 3 isotropic tensors (in 3 dimensions) are given by $\lambda e_{ijk}$ for $\lambda \in \mathbb{R}$. 
Rank 4 isotropic tensors

The most general rank 4 isotropic tensor (in 3 dimensions) is given by $\lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$ for $\lambda, \mu, \nu \in \mathbb{R}$.
Symmetric and Antisymmetric Tensors
100 Symmetric and antisymmetric indices and tensors

Given a rank $k$ tensor $x_{i_1 i_2 \ldots i_k}$. The tensor is called symmetric in the indices $i_m$ and $i_n$ (w.l.o.g. $m < n$) if

$$x_{i_1 \ldots i_{m-1} i_m i_{m+1} \ldots i_{n-1} i_n i_{n+1} \ldots i_k} = x_{i_1 \ldots i_{m-1} i_n i_{m+1} \ldots i_{n-1} i_m i_{n+1} \ldots i_k}$$  \hspace{1cm} (272)

and it is called antisymmetric in the indices $i_m$ and $i_n$ if

$$x_{i_1 \ldots i_{m-1} i_m i_{m+1} \ldots i_{n-1} i_n i_{n+1} \ldots i_k} = -x_{i_1 \ldots i_{m-1} i_n i_{m+1} \ldots i_{n-1} i_m i_{n+1} \ldots i_k} .$$  \hspace{1cm} (273)

If the tensor $x_{i_1 i_2 \ldots i_k}$ is symmetric in any pair of its indices then we call it (totally) symmetric and if it is antisymmetric in any pair of its indices then we call it (totally) antisymmetric.
Example 7.20 \( \delta_{ij} \) is a symmetric rank 2 tensor and \( \epsilon_{ijk} \) is an antisymmetric rank 3 tensor. The inertia tensor is a symmetric rank 2 tensor but the conductivity tensor is not necessarily symmetric; this will depend on the structure of the conducting material. The outer product of two symmetric tensors such as \( s_{ijkl} \) is symmetric in \( i, j \) and in \( k, l \) but not necessarily totally symmetric. The inner product of two symmetric tensors such as \( s_{ij} u_{jk} \) is not necessarily symmetric in any indices.
101 Inner product over symmetric and antisymmetric indices

The (double) inner product over two pairs of indices which involves two symmetric indices for one tensor and two antisymmetric indices for the other tensor equals 0. For example \( \epsilon_{ijk}s_{ij} = 0 \) for the symmetric tensor \( s_{ij} \).
Decomposition of a rank 2 tensor in symmetric and antisymmetric part

Every rank 2 tensor (matrix) $x_{ij}$ can be decomposed uniquely into its symmetric part $s_{ij}$ and its antisymmetric part $a_{ij}$ such that $x_{ij} = s_{ij} + a_{ij}$ with $s_{ij}$ symmetric and $a_{ij}$ antisymmetric. We have $s_{ij} = \frac{1}{2}(x_{ij} + x_{ji})$ and $a_{ij} = \frac{1}{2}(x_{ij} - x_{ji})$. 
30 Examples of Tensors

Quadric Surfaces

Let $S_{ij}$ be an $n$ dimensional symmetric rank 2 tensor. Symmetric matrices can be diagonalised with real eigenvalues using an orthogonal basis transformation. Let $A$ be the diagonal $n$ by $n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (the eigenvalues are not necessarily distinct) and therefore represents this tensor with respect to its eigenbasis.

We construct the quadratic form

$$Q(\mathbf{x}, \mathbf{x}) = x_i S_{ij} x_j,$$  \hspace{1cm} (278)

with $\mathbf{x}$ being an $n$-dimensional vector. The quadratic form $Q$ is obviously a scalar: $Q'(\mathbf{x}', \mathbf{x}') = x_i' S_{ij} x_j' = L_{ik} x_k L_{jm} S_{im} L_{jp} x_p = \delta_{kl} \delta_{mp} x_k S_{im} x_p = x_k S_{km} x_m$. If we choose $x_i'$ to be the components of $\mathbf{x}$ with respect to the eigenbasis of $S_{ij}$ then we find $Q(\mathbf{x}', \mathbf{x}') = \lambda_1 x_1'^2 + \ldots + \lambda_n x_n'^2$ and therefore the surface defined by $Q = \text{constant}$ is a quadric surface. If $S_{ij} = \Theta_{ij}$, the inertia tensor, then the corresponding quadric surface is called the inertia quadric.

In $\mathbb{R}^3$ the quadric surface is for example an ellipsoid for $\lambda_1, \lambda_2, \lambda_3 > 0$, a hyperboloid for $\lambda_1, \lambda_2 > 0, \lambda_3 < 0$ or a cylinder for $\lambda_1 = \lambda_2 > 0$ and $\lambda_3 = 0$. The eigenvectors are called the principal axes of the quadric and the eigenvalues are called the principal values. It is easy to show that the quantities $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(S)$, $\lambda_1 \lambda_2 \lambda_3 = \det(S)$ and $\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$ are invariant under rotation of axes. They are called the principal invariants of the quadric.
Metric Tensor

In example ?? we introduced the metric tensor \( g_{ij} \) contracted with the differential \( dx_i dx_j \) which is the scalar \( ds^2 = g_{ij} dx_i dx_j \), called the line element. In case \( g_{ij} = \delta_{ij} \) then \( ds^2 = g_{ij} dx_i dx_j = dx_i dx_j \). Without loss of generality we can assume \( g_{ij} \) to be symmetric\(^1\). Integrating along a path \( \gamma \) and using that along the path \( dx_i = \frac{d\gamma_i}{dt} \) then the integral \( \int_\gamma \sqrt{ds^2} \) leads to

\[
\int_\gamma \sqrt{ds^2} = \int_\gamma \sqrt{dx_i dx_i} = \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt ,
\]

which is simply the line integral along the path \( \gamma \) and therefore the length of the path \( \gamma \). Therefore \( ds^2 = dx_i dx_i \) simply defines the Euclidean structure of the space \( \mathbb{R}^3 \) (or more generally \( \mathbb{R}^n \)). The Euclidean metric measures distances of two points using the rule of Pythagoras\(^2\). \( ds^2 \) defines the distance between neighbouring points and integrating over \( ds \) along a path defines the length of the path. For the Euclidean metric \( \delta_{ij} \) every non-trivial vector is an eigenvector with respect to the eigenvalue 1. But sometimes we may want to measure lengths and distances in different directions with different weight factors. For example, if the structure of the space is such that it is twice as difficult to move in direction \( x \) than it is to move in direction \( y \) or \( z \) then we may want to use the metric

\[
G = \begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

instead of the Euclidean metric to measure distances in such a space. Generally, the symmetric metric tensor \( g_{ij} \) defines the structure of local distances and the integral \( \int_\gamma \sqrt{g_{ij} dx_i dx_j} \) defines the global distances (i.e. the length of the path \( \gamma \)) in this structure. Using a parametrisation \( \gamma(t) \) we find

\[
\int_\gamma \sqrt{g_{ij} dx_i dx_j} = \int_a^b \sqrt{g_{ij} \frac{d\gamma_i}{dt} \cdot \frac{d\gamma_j}{dt}} \, dt .
\]

The integrals along paths introduced in Sec. ?? obviously used the Euclidean metric which is generated by the 3 by 3 unit matrix. In contrast, in Einstein’s\(^3\) Theory of Special Relativity distances in space-time are measured by the non-Euclidean line element \( ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \). This metric is called the Minkowski metric. For this metric, obviously, \( ds^2 \) can be negative. If for two points in space-time \( c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 < 0 \) then the distance is called space-like, if it is \( > 0 \) then it is called time-like and if it is \( = 0 \) then it is called light-like or null since light moves on light-like paths, whilst we are restricted to move on time-like paths!

\(^1:\) Due to the symmetry of \( dx_i dx_j \), any antisymmetric part of \( g_{ij} \) would anyway disappear in the inner product \( g_{ij} dx_i dx_j \) according to \([101]\). Therefore, we might as well assume that \( g_{ij} \) does not have an antisymmetric part.

\(^2:\) Pythagoras of Samos, about 569 BC-475 BC.

\(^3:\) Albert Einstein, 1879-1955.

\(^4:\) Hermann Minkowski, 1864-1909.
Acknowledgements

I am extremely grateful for the many comments and corrections I have received from the students attending my lectures on Vector Calculus in Lent 2006 and I would very much welcome to receive any further corrections and comments for improvements (md131@cam.ac.uk). Good luck to all of you for your examinations in June!