1 The functions $u$ and $w$ are defined in a volume $\mathcal{V}$. Show that

$$\int_{\mathcal{V}} \|
abla w\|^2 dV - \int_{\mathcal{V}} \|
abla u\|^2 dV = \int_{\mathcal{V}} \|\nabla (w - u)\|^2 dV + 2 \int_{\mathcal{V}} \nabla u \cdot \nabla (w - u) dV.$$ 

If $u = w$ on the boundary of $\mathcal{V}$ and $u$ is harmonic, show that

$$\int_{\mathcal{V}} \|
abla w\|^2 dV \geq \int_{\mathcal{V}} \|
abla u\|^2 dV.$$ 

2 Find, by three different methods, the gravitational field everywhere due to a spherical shell with density given by

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } b < r < \infty. \end{cases}$$

where $\rho_0$ is a constant. The methods referred to are: direct solution of Poisson’s equation; use of Gauss's flux theorem; use of the integral form of the general solution of Poisson’s equation. You should assume that the potential is a function only of $r$ and that the potential and its first derivative are continuous at $r = a$ and $r = b$. What is the justification for these assumptions?

3 The scalar field $\varphi$ is harmonic in a volume $\mathcal{V}$ bounded by a closed surface $\mathcal{S}$. Given that $\mathcal{V}$ does not contain the origin $(r = 0)$, show that

$$\int_{\mathcal{S}} \left( \varphi \nabla \left( \frac{1}{r} \right) - \left( \frac{1}{r} \right) \nabla \varphi \right) \cdot d\mathbf{S} = 0.$$ 

Now let $\mathcal{V}$ be the volume given by $\epsilon \leq r \leq a$ and let $\mathcal{S}_1$ be the surface $r = a$. Given that $\varphi(\mathbf{x})$ is harmonic for $r \leq a$, use the above result, in the limit $\epsilon \to 0$, to show that

$$\varphi(0) = \frac{1}{4\pi a^2} \int_{\mathcal{S}_1} \varphi(\mathbf{x}) \, dS.$$ 

Deduce that if $\varphi$ is harmonic (but not constant) in a general volume $\mathcal{V}$, then it attains its maximum and minimum values on $\mathcal{S}$.

4 If $\nabla^2 \varphi = \rho$ in a volume $\mathcal{V}$ enclosed by $\mathcal{S}$ and $\mathbf{x}_0$ is a point within $\mathcal{V}$, show that

$$4\pi \varphi(\mathbf{x}_0) = -\int_{\mathcal{V}} \frac{\rho(\mathbf{x})}{||\mathbf{x} - \mathbf{x}_0||} \, dV + \int_{\mathcal{S}} \left( \frac{1}{||\mathbf{x} - \mathbf{x}_0||} \frac{\partial \varphi}{\partial \mathbf{n}}(\mathbf{x}) - \varphi(\mathbf{x}) \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{||\mathbf{x} - \mathbf{x}_0||} \right) \right) \, dS.$$ 

5 A physical entity is represented in each Cartesian frame by the array of numbers $\delta_{ij}$ $(i, j = 1, 2, 3)$, equal to 1 when $i = j$ and 0 when $i \neq j$. Show that this entity is a tensor. What is meant by saying that $\delta_{ij}$ is an isotropic tensor?
6 Three Cartesian frames of reference in $\mathbb{R}^3$ are such that the $i^{th}$ axis of the first frame coincides with the $(i + n)^{th}$ axis (modulo 3) of the $(n + 1)^{th}$ frame $(n = 0, 1, 2)$.

(i) A physical entity has components

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, in the three frames respectively. Show that this entity cannot be a tensor.

(ii) Show that the entity with respective components

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, could be a tensor, and on the assumption that it is, find its components in an arbitrary Cartesian frame whose axes are at angles $\phi_1, \phi_2$ and $\phi_3$ to the 1-axis of the first frame with $\cos \phi_1 = \lambda, \cos \phi_2 = \mu$ and $\cos \phi_3 = \nu$ (i.e. the axes have direction cosines $(\lambda, \mu, \nu)$ with respect to the 1-axis of the first frame).

7 The conductivity tensor $\sigma_{ij}$ of a homogeneous anisotropic material has components

$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

with respect to a set of Cartesian axes. Show that no current can flow in the direction $(\sqrt{2}, -1, 1)$. Show also that the direction in which the conductivity is greatest is $(\sqrt{2}, 3, 1)$, and that the conductivity in that direction is 4.

8 (i) Show that if the array of numbers $A_{ij}$ represents a tensor, then so does $A_{ji}$.

(ii) The tensor $A_{ij}$ is symmetric or antisymmetric in $i, j$ ($A_{ij} = \pm A_{ji}$). Show that the components in another Cartesian frame also satisfy $A'_{ij} = \pm A'_{ji}$ [i.e. symmetry and antisymmetry of the component array are tensor properties, being preserved under rotation of the coordinate axes].

9 Given a non-zero vector $v_i$, the orthogonal projection tensor $P_{ij}$ is defined by

$P_{ij} = \delta_{ij} - \frac{v_i v_j}{v_k v_k}$.

(i) Verify that $P_{ij}$ satisfies (a) $P_{ij} v_j = 0$ and (b) $P_{ij} u_j = u_i$ for any vector $u_i$ which is orthogonal to $v_i$.

(ii) Show that $P_{ij}$ is unique, that is, if another tensor $T_{ij}$ satisfies both (a) and (b), then $(P_{ij} - T_{ij}) w_j = 0$ for any vector $w_i$.

(iii) For $A_{ij} = \epsilon_{ijk} v_k$, show that $P_{ij} A_{jk} A_{km} = -v_k v_k P_{km}$.

10 If $A_{ijk}(x)$ is a tensor of rank 3, show that $\partial A_{ijk}/\partial x_k$ and $\partial A_{ijk}/\partial x_l$ are tensors of rank 2 and 4 respectively.
11 The array $D_{ikm}$ with $\mathbb{R}^3$ elements is not known to represent a tensor. If, for every symmetric tensor represented by $a_{km}$,

$$\mathbf{b}_k = D_{ikm}a_{km}$$

represents a vector, what can be said about the transformation properties under rotations of the coordinate axes of

(i) $D_{ikm}$,  
(ii) $D_{ikm} + D_{ink}$?

12 Evaluate the following integrals over the whole of $\mathbb{R}^3$ for $\gamma$ a positive constant and $r^2 = x_p x_p'$:

(i) $\int r^{-4} e^{-\gamma r^2} x_i x_j dV$,  
(ii) $\int r^{-5} e^{-\gamma r^2} x_i x_j x_k dV$.

[Hint: use the fact that the integrals are isotropic tensors.]

13 (i) For any tensor $T_{ik}$ in $\mathbb{R}^3$, prove directly from the transformation property of a tensor that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ik} T_{ki}, \quad \gamma = T_{ik} T_{km} T_{mi}$$

are invariant under rotation of the coordinate axes.

(ii) If $T_{ik}$ is a symmetric tensor, express these invariants in terms of its eigenvalues. Deduce that the cubic equation for the eigenvalues $\lambda$ is

$$\lambda^3 - \alpha \lambda^2 + \frac{1}{2}(\alpha^2 - \beta) \lambda - \frac{1}{6}(\alpha^3 - 3\alpha \beta + 2\gamma) = 0.$$ (iii) Given that the most general isotropic rank $4$ tensor is $\lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$ for $\lambda, \mu, \nu \in \mathbb{R}$ show that

$$\epsilon_{ijk} \epsilon_{\ell mn} = \delta_{ij} \delta_{km} - \delta_{im} \delta_{jk}.$$

14 For the rigid body $x^2 + y^2 \leq z^2$, $0 \leq z \leq 1$ with mass density $\rho(x) = 1$ find the inertia tensor relative to the origin and find its principal axes. Use the parallel axis theorem to find the inertia tensor about the point $(0,0,1)$.

15 A conductor positioned in a magnetic field $\mathbf{H}$ carries a steady current density $\mathbf{j} = \text{curl} \mathbf{H}$, and the magnetic flux intensity $\mathbf{B} = \mu \mathbf{H}$ satisfies $\text{div} \mathbf{B} = 0$. The mechanical force per unit volume acting on the conductor can be written $\mathbf{j} \times \mathbf{B}$. If the permeability $\mu$ is a constant, show that this force per unit volume can be written as $\partial s_{ik}/\partial x_k$ in terms of a tensor

$$s_{ik} = \mu (H_i H_k - \frac{1}{2} H_m H_m \delta_{ik}).$$

I would appreciate any comments and corrections from students and supervisors. Please e-mail md131@cam.ac.uk.