1 (i) Let $\psi(x)$ be a scalar field and $\mathbf{v}(x)$ a vector field. Show, using suffix notation, that
\[
\nabla \cdot (\psi \mathbf{v}) = (\mathbf{v} \cdot \nabla) \psi + \psi \nabla \cdot \mathbf{v}, \quad \nabla \times (\psi \mathbf{v}) = (\nabla \psi) \times \mathbf{v} + \psi \nabla \times \mathbf{v}.
\]
(ii) Evaluate (using suffix notation where necessary) the divergence and the curl of the following:
\[
r \times, \quad a(x, b), \quad \mathbf{a} \times \mathbf{x}, \quad \frac{\mathbf{x} - \mathbf{a}}{||\mathbf{x} - \mathbf{a}||^2},
\]
where $r = ||x||$, and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.
(iii) Let $\psi(x)$ be a scalar field and $\mathbf{v}(x)$ a vector field. Show that
\[
curl(\text{grad}(\psi)) = 0, \quad \text{div}(\text{curl}(\mathbf{v})) = 0, \quad \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}
\]
(iv) Use suffix notation and summation convention to show that $\nabla r^n = nr^{n-2} \mathbf{x}$, where $r = ||x||$. Obtain the same results using plane polar coordinates.
In plane polar coordinates, for a function $f$ of $r$ and $\phi$ only, $\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi$.

2 Show that the unit basis vectors of cylindrical polar coordinates satisfy
\[
\frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho, \quad \text{(s)}
\]
all other derivatives of the three basis vectors being zero.
Given that the gradient operator in cylindrical polars is
\[
\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z},
\]
use (s) to obtain expressions for $\nabla \mathbf{A}$ and $\nabla \times \mathbf{A}$, where $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$.

3 Verify Stokes' theorem for the open hemispherical surface $r = 1, \ z \geq 0$, and the vector field
\[
\mathbf{F}(x) = (y, \ -x, \ z).
\]

4 A fluid flow has the velocity vector $(0, \ 0, \ z+a)$ in cartesian coordinates, where $a$ is a constant. Calculate the volume flux of fluid flowing across the open hemispherical surface $r = a, \ z \geq 0$, and also the volume flowing across the disc $z = 0, \ r \leq a$. Verify that the divergence theorem holds.

5 Let $\mathbf{F}(x) = (x^3 + 3y + z^2, \ y^3, \ x^2 + y^2 + 3z^2)$, and let $S$ be the open surface
\[
1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.
\]
Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\int_S \mathbf{F} \cdot dS$.
Verify your result by calculating the integral directly.

1
6 By applying Stokes’ theorem to the vector field \(\mathbf{k} \times \mathbf{F}\), where \(\mathbf{k}\) is an arbitrary constant vector and \(\mathbf{F}(\mathbf{x})\) is a vector field, show that

\[
\int_{\partial S} d\mathbf{x} \times \mathbf{F} = \int_{S} (d\mathbf{S} \times \nabla) \times \mathbf{F},
\]

where the curve \(\partial S\) bounds the open surface \(S\).

Verify this result when \(\partial S\) is the unit square in the \(xy\)-plane with opposite vertices at \((0,0,0)\) and \((1,1,0)\) and \(\mathbf{F}(\mathbf{x}) = \mathbf{x}\).

7 By applying the divergence theorem to the vector field \(\mathbf{k} \times \mathbf{A}\), where \(\mathbf{k}\) is an arbitrary constant vector and \(\mathbf{A}(\mathbf{x})\) is a vector field, show that

\[
\int_{V} \nabla \times \mathbf{A} \, dV = -\int_{\partial V} \mathbf{A} \times d\mathbf{S},
\]

where the surface \(\partial V\) encloses volume \(V\).

Verify this result when \(\partial V\) is the sphere \(\|\mathbf{x}\| = R\) and \(\mathbf{A} = (z,0,0)\) in cartesian coordinates.

8* (i) Let \(S\) be a flat surface in the \(xy\)-plane. Show that Green’s theorem in the plane can be obtained by applying Stokes’ theorem to the vector field \(\mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}), 0)\) and the surface \(S\).

(ii) For a (differentiable) function \(f(x) : [a,b] \rightarrow \mathbb{R}\) define the vector field \(\mathbf{F}(\mathbf{x}) = (0, f(x), 0)\). Apply Stokes’ theorem to the vector field \(\mathbf{F}\) on the rectangle \(a \leq x \leq b\) and \(0 \leq y \leq 1\). Comment on your result.

9 Let

\[
I = \int_{S} \frac{x \, dS}{\|x\|^3}.
\]

Show that \(I = 4\pi\) if \(S\) is the sphere \(\|x\| = R\) and that \(I = 0\) if \(S\) bounds a volume that does not contain the origin \((x = 0)\).

Show that the electric field, defined for \(x \neq a\) by \(\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi \varepsilon_0 \|x - a\|^3}\), satisfies

\[
\int_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \begin{cases} 
0 & \text{if } a \notin V \\
\frac{q}{\varepsilon_0} & \text{if } a \in V 
\end{cases}
\]

where \(\partial V\) is a closed surface bounding a volume \(V\) and \(a \notin \partial V\), and where the electric charge, \(q\), and permittivity of free space, \(\varepsilon_0\), are constants. This is Gauss’s law for a point electric charge.

10 The vector field \(\mathbf{B}(\mathbf{x})\) is given in cylindrical polar coordinates \(\rho, \phi\) and \(z\) (for \(\rho \neq 0\)) by

\[
\mathbf{B}(\mathbf{x}) = \rho^{-1} \hat{\mathbf{e}}_\phi.
\]

Evaluate \(\nabla \times \mathbf{B}\) using the formula for curl in cylindrical polar coordinates. Calculate \(\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{x}\), where \(\mathcal{C}\) is the circle \(z = 0\), \(\rho = 1\) and \(0 \leq \phi \leq 2\pi\). Does Stokes’s theorem apply? Why not?
11 The surface $S$ encloses a volume in which the scalar field $\varphi$ satisfies the Klein-Gordon equation
\[ \nabla^2 \varphi = m^2 \varphi, \]
where $m$ is a real non-zero constant. Prove that $\varphi$ is uniquely determined if either $\varphi$ or $\partial \varphi / \partial n$ is given on $S$.

12 Find all solutions of the two-dimensional Laplace equation $\nabla^2 f = 0$ that can be written in the separable form $f(r, \phi) = R(r) \Phi(\phi)$, where $r$ and $\phi$ are plane polar coordinates.

Hence solve, for $r < a$, the following boundary value problem, assuming that $f(r, \phi)$ satisfies a reasonable physical condition at $r = 0$:
\[ \nabla^2 f = 0, \quad f(a, \phi) = \sin \phi. \]
Find also the solution for $r > a$ that satisfies $f(r, \phi) \to 0$ as $r \to \infty$.

13 The scalar function $\varphi$ is a function only of the radial coordinate $r$ in $\mathbb{R}^3$. Use Cartesian co-ordinates and the chain rule to show that
\[ \nabla \varphi = \varphi'(r) \frac{\mathbf{x}}{r}, \quad \nabla^2 \varphi = \varphi''(r) + \frac{2}{r} \varphi'(r). \]
Find the solution of $\nabla^2 \varphi = 1$ in the region $r \leq a$ that is bounded and satisfies $\varphi(a) = 1$.

14 Show that, within a closed surface $S$, not more than one solution of Poisson’s equation satisfies the boundary condition
\[ \frac{\partial \varphi}{\partial n} + \varphi = 0 \]
on $S$, where $g(x) \geq 0$ on $S$.

Show that $\varphi(x) = x$ satisfies Laplace’s equation and the above boundary condition with $S$ being the unit sphere $\|x\| = 1$. Deduce that the condition $g(x) \geq 0$ on $S$ cannot be omitted in the above uniqueness theorem.

I would appreciate any comments and corrections from students and supervisors. Please e-mail md131@cam.ac.uk.