Vector Calculus, Examples sheet 1 | Dr M. Dörrzapf

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1 The curve parametrised by $\boldsymbol{\gamma}(t) = (a\cos^3 t, a\sin^3 t)$, $0 \le t \le 2\pi$, $a \in \mathbb{R}^+$ is called *Astroid*. Find its length.

2 The curve defined by $y^2 = x^3$ is called *Neile's parabola*, named after William Neil (1637-1670). Find the length of the segment of Neil's parabola with $0 \le x \le 4$.

3 In $\mathbb{R}^2$ a path is defined in polar coordinates by $r = f(\phi)$, $a \le \phi \le b$ with some function f. Show that the length of the path is

$$L = \int_a^b \sqrt{(f(\phi))^2 + (f'(\phi))^2}\,\mathrm{d}\phi\,.$$

Sketch the paths (i) $r = a\phi$, $0 \le \phi \le 2\pi$, $a > 0$ (*Archimedes' Spiral*) and (ii) $r = a(1+\cos\phi)$, $0 \le \phi \le 2\pi$, $a > 0$ (*Cardioid*) and calculate their lengths.

4 (i) A circular helix is given by

$$\mathbf{x} = (a\cos t, a\sin t, ct)\ ,$$

where $a, c \in \mathbb{R}^+$. Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{p}$, binormal $\mathbf{b}$, and torsion τ.

(ii)* Explain generally why the tangent $\mathbf{t}$, the principal normal $\mathbf{p}$ and the binormal $\mathbf{b}$ form an orthonormal system. Show that the torsion can be written as

$$\tau = \frac{[\mathbf{t}, \frac{\partial \mathbf{t}}{\partial s}, \frac{\partial^2 \mathbf{t}}{\partial s^2}]}{\kappa^2},$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the scalar triple product $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$. Verify this identity for the circular helix in part (i).

5 * A sculpture in the form of a conical helix is described by the path $\mathbf{x}(t) = (t\cos t, t\sin t, t)$, $2\pi \le t \le 4\pi$. The sculpture is made out of an inhomogeneous material which has a line density of μ at the starting point $t = 2\pi$ and increases linearly in t to 2μ at $t = 4\pi$. Find the mass of this sculpture.

6 Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)\,,$$

along (i) the straight line path joining the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t$, $y = t$, $z = t^2$ with $0 \leqslant t \leqslant 1$.

7 The vector force fields $\mathbf{F}$ and $\mathbf{G}$ are defined by $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$ respectively. (i) Compute the line integrals $\int \mathbf{F}.\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}.\mathrm{d}\mathbf{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. (ii) Compute the line integrals $\int \mathbf{F}.\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}.\mathrm{d}\mathbf{x}$ along the path $\mathbf{x}(t) = (t, t^2, t^2)$ from $(0,0,0)$ to $(1,1,1)$.

8 Sketch the graph of the function $f(x,y) = \cos(|x| + |y|)$ in the region $|x| + |y| \leq \frac{\pi}{2}$ and find the volume underneath this graph.

9 The closed curve $\mathcal{C}$ consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The region enclosed by $\mathcal{C}$ is $\mathcal{S}$. Show, by calculating the integrals explicitly, that

$$\int_{\mathcal{C}} (xy^2\mathrm{d}y + x^2y\,\mathrm{d}x) = \int_{\mathcal{S}} (y^2 - x^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{104}{105}a^4.$$

where $\mathcal{C}$ is described anticlockwise.

10 Show (without changing the order of integration) that

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y\right]\mathrm{d}x = \frac{1}{2} \qquad \text{and} \qquad \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}x\right]\mathrm{d}y = -\frac{1}{2}$$

Comment on these results.

11 A solid cone is bounded by the surface $\theta = \alpha$ in spherical polar coordinates and the surface $z = a$. Its mass density is $\rho_0 \cos\theta$. By evaluating a volume integral find the mass of the cone.

12 Use the substitution $x = \rho\cos\theta$, $y = \frac{1}{2}\rho\sin\theta$, to evaluate

$$\int_{\mathcal{S}} \frac{x^2}{x^2 + 4y^2}\,\mathrm{d}S \quad ,$$

where $\mathcal{S}$ is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

13 The region $\mathcal{D}$ is bounded by the segments $x = 0$, $0 \leqslant y \leqslant 1$; $y = 0$, $0 \leqslant x \leqslant 1$; $y = 1$, $0 \leqslant x \leqslant \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the (x,y) plane from the (u,v) plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch $\mathcal{D}$ and also the two regions in the (u,v) plane which are mapped into it. Hence evaluate

$$\int_{\mathcal{D}} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{1}{2}}} .$$

I would appreciate any comments and corrections from students and supervisors. Please e-mail `md131@cam.ac.uk`.