VI LAPLACE'S EQUATION

19 Laplace's equation

As an application of the theory of multidimensional integration developed in the previous chapters we will now study the solutions of a particular second order partial differential equation: the *Laplace equation*

$$\nabla^2\Phi = 0 , \tag{170}$$

and the corresponding inhomogeneous equation: the *Poisson equation*[a]

$$\nabla^2\Phi = f(\mathbf{x}) . \tag{171}$$

Both differential equations are particularly important in many areas of mathematics and physics. If Φ is for example the potential of the electric field $\mathbf{E}$ such that $\mathbf{E} = -\nabla\Phi$ then Φ satisfies Poisson's equation

$$\nabla^2\Phi = -\frac{\rho(\mathbf{x})}{\epsilon_0} , \tag{172}$$

where $\rho(\mathbf{x})$ is the charge density and ϵ_0 is the permittivity of free space. If we replace $\nabla\Phi$ in $\nabla^2\Phi$ by $\mathbf{E}$ then Poisson's equation can be re-written as

$$\nabla.\mathbf{E}(\mathbf{x}) = 4\pi\rho(\mathbf{x}) . \tag{173}$$

Considering that the divergence $\nabla.\mathbf{E}$ of the vector field describes where the vector field has sinks or sources EQ. 173 simply says that the electric field $\mathbf{E}$ starts or ends (i.e. has sources or sinks) only at points where there are charges (i.e. the charge density $\rho(\mathbf{x}) \neq 0$).

Similarly to EQ. 172 the potential of the gravitational field $\mathbf{G}$ satisfies

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}) , \tag{174}$$

where $\rho(\mathbf{x})$ is the mass density and G is the gravitational constant and therefore the gravitational field starts (i.e. has sources) only at points where there is mass.

Using EQ. 155 we can easily see that $\frac{-1}{r}$ is a solution of Laplace's equation in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. The corresponding vector field $\nabla\Phi = \frac{\mathbf{x}}{r^3}$ is the divergence free vector field of a point charge which we have already considered in EQ. 42.

Poisson's equation is obviously a linear differential equation and Laplace's equation is the corresponding homogeneous equation. Due to the linearity of Poisson's equation the most general solution to Poisson's equation Φ is given by a *particular solution* to Poisson's equation Φ_p plus the most general homogeneous solution, which is the most general solution to Laplace's equation Ψ. In order to prove this statement we need to show two things: first $\Phi_p + \Psi$ are solutions to Poisson's equation and secondly there are no others. From $\nabla(\Phi_p + \Psi) = \nabla\Phi_p + \nabla\Psi = f(\mathbf{x}) + 0$ it is obvious that $\Phi_p + \Psi$ is a solution of Poisson's equation. Conversely, if we assume that there is a solution $\hat{\Phi}$ of EQ. 171 which is not of the form $\Phi_p + \Psi$ then $\hat{\Phi} - \Phi_p$ would obviously satisfy $\nabla(\hat{\Phi} - \Phi_p) = f(\mathbf{x}) - f(\mathbf{x}) = 0$ and therefore $\hat{\Phi} - \Phi_p$ would be a solution

[a] Simon-Denis Poisson, 1781-1840.

of Laplace's equation and is consequently included in the most general solution Ψ of Laplace's equation. But this means that $\hat{\Phi} = \Phi_p + \Psi_p$ for some Ψ_p contained in Ψ which contradicts the assumption that $\hat{\Phi}$ is not of the form $\Phi_p + \Psi$.

Since the general solution of Poisson's equation is a particular solution plus the general solution to Laplace's equation means that is now first important to analyse the general solution of Laplace's equation. Let us therefore at first concentrate on solving Laplace's equation and we will come back to Poisson's equation in SEC. 23. To start with we look at the Laplace's equation in two dimensions $\frac{\partial^2\Phi(x,y)}{\partial x^2} + \frac{\partial^2\Phi(x,y)}{\partial y^2} = 0$ and transform the Laplacian into plane polar coordinates as in EQ. 148. The Laplace equation in plane polar co-ordinates is therefore given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi(r,\phi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi(r,\phi)}{\partial\phi^2} = 0 . \tag{175}$$

We first try to find a particular type of solutions of the form $\Phi(r,\phi) = R(r)P(\phi)$. At this stage we of course do not have any guarantee that solutions of this type exists. If such solutions do exist then we call them *separable solutions*. Substituting $\Phi(r,\phi) = R(r)P(\phi)$ into EQ. 175 leads after re-arranging to

$$\frac{r}{R(r)}\frac{\partial}{\partial r}(rR'(r)) = -\frac{P''(\phi)}{P(\phi)} . \tag{176}$$

The left hand side of EQ. 176 is a function of r only and the right hand side is only a function of ϕ we therefore find that the left hand side and the right hand side of EQ. 176 have to be equal to a constant $k \in \mathbb{R}$. We therefore find two separated differential equations

$$P''(\phi) = -kP(\phi) , \tag{177}$$

$$\frac{r}{R(r)}\frac{\partial}{\partial r}(rR'(r)) = k . \tag{178}$$

EQ. 178 has three possible types of solutions

$$P(\phi) = a\cos(\sqrt{k}\phi) + b\sin(\sqrt{k}\phi) \quad , \quad k > 0 , \tag{179}$$

$$P(\phi) = a + bt \quad , \quad k = 0 , \tag{180}$$

$$P(\phi) = ae^{\sqrt{-k}\phi} + be^{-\sqrt{-k}\phi} \quad , \quad k < 0 , \tag{181}$$

but considering that continuity of $P(\phi)$ requires $P(0) = P(2\pi)$ only the first of these three solutions can lead to non-trivial solutions. This also means that $k > 0$. Furthermore $P(\phi) = P(\phi + 2\pi)$ for any ϕ requires that $\sqrt{k} = n$ for some $n \in \mathbb{N}$. Therefore the only possible solutions for $P(\phi)$ are the solutions $P_n(\phi) = a\cos(n\phi) + b\sin(n\phi)$ with $k = n^2$. Substituting this back into equation EQ. 178 for the radial component $R(r)$ leads to

$$r\frac{\partial}{\partial r}(rR'(r)) = n^2 R(r) . \tag{182}$$

Using the trial function $R(r) = cr^\alpha$ we can easily find the general solution $cr^n + dr^{-n}$. And therefore we have found separable solutions of Laplace's equation

$$\Phi_n(r,\phi) = (cr^n + dr^{-n})(a\cos(n\phi) + b\sin(n\phi)) , \tag{183}$$

for each $n \in \mathbb{N}$. It is clear that any linear combination of these separable solutions are again solutions of Laplace's equation (which are in fact not separable). We have therefore found infinitely many solutions to Laplace's equation. It can be shown that any solution to Laplace's equation can be written as a series expansion in terms of separable solutions:

$$\Phi(\mathbf{x}) = \sum_{n=0}^{\infty}(c_n r^n + d_n r^{-n})(a_n \cos(n\phi) + b_n \sin(n\phi)) \; . \qquad (184)$$

This will be considered further in the Part IB course *Methods*. Ultimately we want to find solutions of Laplace's equation satisfying certain boundary conditions. For example if we were looking for a solution Φ of the two-dimensional Laplace equation which satisfies that $\Phi(\mathbf{0}) = 0$ and $\Phi = \cos\phi$ on the unit circle with ϕ being the angle of the standard plane polar co-ordinates then our earlier considerations show that the separable solution $\Phi(r, \phi) = r\cos\phi$ is a solution to this *boundary value problem*. But whether this is the unique solution to this boundary value problem remains to be investigated which will be done in SEC. 21. In exactly the same way we could look for separable solutions of the three dimensional Laplace equations using EQ. 155. Laplace's equation can of course equally well be separated in Cartesian co-ordinates. Which co-ordinates are most suitable for the separation depends mainly on what is most suitable to describe the boundary conditions.

We will now study the boundary value problem of the Laplace equation in three dimensions and then later the boundary value problem of the Poisson equation in more detail. But before we continue we will need to rewrite the divergence theorem in a form which allows us to study the properties of the Laplace operator $\boldsymbol{\nabla}^2$ more easily.

20 Green's theorems

If we are given two real valued functions (which we want to call *scalar fields*) ϕ and ψ defined on some domain G then we can construct a vector field $\mathbf{F} := \phi\boldsymbol{\nabla}\psi$. The divergence of $\mathbf{F}$ is then given by

$$\boldsymbol{\nabla}.\mathbf{F} = \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\psi + \phi\boldsymbol{\nabla}^2\psi \; . \qquad (185)$$

We can now use the divergence theorem on $\boldsymbol{\nabla}.\mathbf{F}$ and obtain for any volume $\mathcal{V}$ which is contained in G that $\int_{\mathcal{V}} \boldsymbol{\nabla}.\mathbf{F} dV = \oint_{\partial V} \phi\boldsymbol{\nabla}\psi.d\mathbf{S}$. If we define $\mathbf{n}$ to be the normalised outward normal to the surface $\partial\mathcal{V}$ then $d\mathbf{S} = \mathbf{n}dS$. Using the directional derivatives theorem [40] we can write $\boldsymbol{\nabla}\psi.d\mathbf{S}$ as $\frac{\partial\psi}{\partial\mathbf{n}}dS$ where $\frac{\partial\psi}{\partial\mathbf{n}}$ is the directional derivative of ψ in direction $\mathbf{n}$. We obtain *Green's first theorem*:

66 Green's first theorem
The scalar fields $\phi, \psi : G \subset \mathbb{R}^3 \to \mathbb{R}$ satisfy for any volume $\mathcal{V} \subset G$

$$\begin{aligned}\int_{\mathcal{V}} \left(\phi\boldsymbol{\nabla}^2\psi + \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\psi\right) dV &= \oint_{\partial V} \phi\boldsymbol{\nabla}\psi.d\mathbf{S} \\ &= \oint_{\partial\mathcal{V}} \phi\boldsymbol{\nabla}\psi.\mathbf{n}dS = \oint_{\partial\mathcal{V}} \phi\frac{\partial\psi}{\partial\mathbf{n}}dS \; , \end{aligned} \qquad (186)$$

where $\mathbf{n}$ is the normalised outward normal vector to the surface $\partial\mathcal{V}$ such that $d\mathbf{S} = \mathbf{n}dS$.

Green's first theorem can be regarded as *integration by parts* for three dimensional volume integrals in the sense that $\int_{\mathcal{V}} \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\psi \, dV$ can be written as a partially integrated two dimensional surface integral for which $\boldsymbol{\nabla}\phi$ has been integrated minus a three dimensional volume integral over the integrated ϕ times the twice differentiated $\boldsymbol{\nabla}^2\psi$:

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\phi . \boldsymbol{\nabla}\psi dV = \oint_{\partial\mathcal{V}} \phi\boldsymbol{\nabla}\psi . d\mathbf{S} - \int_{\mathcal{V}} \phi\boldsymbol{\nabla}^2\psi dV . \tag{187}$$

If we write down Green's first theorem again but now with ϕ and ψ exchanged and then subtract this equation from the original EQ. 186 we obtain *Green's second theorem.*

67 Green's second theorem
The scalar fields $\phi, \psi : G \subset \mathbb{R}^3 \to \mathbb{R}$ satisfy for any volume $\mathcal{V} \subset G$

$$\int_{\mathcal{V}} \left(\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi\right) dV = \oint_{\partial\mathcal{V}} \left(\phi\frac{\partial\psi}{\partial\mathbf{n}} - \psi\frac{\partial\phi}{\partial\mathbf{n}}\right) dS , \tag{188}$$

where $\mathbf{n}$ is the normalised outward normal vector to the surface $\partial\mathcal{V}$ such that $d\mathbf{S} = \mathbf{n}dS$.

21 Harmonic functions

A solution to Laplace's equation is called a *harmonic function.* In SEC. 19 we have given examples of harmonic functions and in particular we have shown that using linear combinations of separable solutions we can construct other solutions.

68 Harmonic functions
A function ϕ, is called harmonic on the volume $\mathcal{V}$ if it satisfies Laplace's equation on $\mathcal{V}$: $\boldsymbol{\nabla}^2\phi(\mathbf{x}) = 0 \quad \forall\mathbf{x} \in \mathcal{V}$.

Let us assume that ψ is harmonic on some volume $\mathcal{V}$. We can then apply Green's first theorem and set $\phi = \psi$ to obtain

$$\int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV = \oint_{\partial\mathcal{V}} \psi\frac{\partial\psi}{\partial\mathbf{n}} dS . \tag{189}$$

In case the boundary condition is such that $\psi(\mathbf{x}) = 0$ on the whole boundary $\partial\mathcal{V}$ then obviously $\oint_{\partial\mathcal{V}} \psi\frac{\partial\psi}{\partial\mathbf{n}} dS = 0$. This means that $\int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV = 0$ and since the continuous function $\|\boldsymbol{\nabla}\psi\|^2 \geq 0$ we have $\|\boldsymbol{\nabla}\psi\|^2 = 0 \ \forall\mathbf{x} \in \mathcal{V}$ and therefore $\psi(\mathbf{x}) = \mathit{constant} \ \forall\mathbf{x} \in \mathcal{V}$. But since $\psi(\mathbf{x}) = 0$ on the boundary $\partial\mathcal{V}$ we find that the *constant* is actually 0 and therefore $\psi(\mathbf{x}) = 0 \ \forall\mathbf{x} \in \mathcal{V}$.

69 Harmonic functions with trivial boundary conditions
The function ψ is harmonic on the volume $\mathcal{V} \subset \mathbb{R}^3$. If $\psi(\mathbf{x}) = 0 \ \forall\mathbf{x} \in \partial\mathcal{V}$ then

$$\psi(\mathbf{x}) = 0 \qquad \forall\mathbf{x} \in \mathcal{V} . \tag{190}$$

In case we have two harmonic functions ϕ_1 and ϕ_2 on some volume $\mathcal{V}$ with the same values on $\partial\mathcal{V}$, we then find that $\psi := \phi_1 - \phi_2$ satisfies $\psi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathcal{V}$ and therefore $\psi = 0$ on the whole of $\mathcal{V}$ according to [69]. Boundary conditions which fix the value of a harmonic function on the boundary of a volume $\mathcal{V}$ are called *Dirichlet boundary conditions*[b].

[b]Johann Peter Gustav *Lejeune* Dirichlet, 1805-1859.

70 Uniqueness theorem for harmonic functions with Dirichlet boundary conditions
The functions ϕ_1 and ϕ_2 are harmonic on the volume $\mathcal{V}$ and satisfy the same Dirichlet boundary conditions $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{V}$ then

$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{V} . \tag{191}$$

Since $\psi(\mathbf{x}) = c$ for $c \in \mathbb{R}$ is harmonic on any volume $\mathcal{V}$ and satisfies the constant Dirichlet boundary conditions $\psi(\mathbf{x}) = c$ on $\partial\mathcal{V}$ we therefore immediately obtain from [70] that a harmonic function which is constant on the boundary $\partial\mathcal{V}$ has to be constant on the whole of $\mathcal{V}$.

71 Constant harmonic functions
The function ψ is harmonic on the volume $\mathcal{V}$. If $\psi(\mathbf{x}) = c \ \forall \mathbf{x} \in \partial\mathcal{V}$, $c \in \mathbb{R}$ then

$$\psi(\mathbf{x}) = c \quad \forall \mathbf{x} \in \mathcal{V} . \tag{192}$$

There are other types of boundary conditions which are important in mathematics and physics:

72 Boundary conditions
A real valued function ψ defined on the volume $\mathcal{V}$ is said to satisfy

(i) Dirichlet boundary conditions if $\psi(x)$ is given as a function on the boundary $\partial\mathcal{V}$;

(ii) Neumann[c] boundary conditions if the directional derivative $\frac{\partial\psi}{\partial\mathbf{n}}$ is given on $\partial\mathcal{V}$;

(iii) boundary conditions of the mixed type if the combination $\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{n}} + f(\mathbf{x})\psi(\mathbf{x})$ is given on the $\partial\mathcal{V}$.

By using Green's first theorem we can now also show in an almost identical way as for [70] that Neumann boundary conditions fix a harmonic function up to a constant.

73 Uniqueness theorem for Neumann boundary conditions
The functions ϕ_1 and ϕ_2 are harmonic on the volume $\mathcal{V}$ and satisfy the same Neumann boundary conditions $\frac{\partial\phi_1(\mathbf{x})}{\partial\mathbf{n}} = \frac{\partial\phi_2(\mathbf{x})}{\partial\mathbf{n}}$ for all $\mathbf{x} \in \partial\mathcal{V}$ then

$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) + c \quad \forall \mathbf{x} \in \mathcal{V} , \tag{193}$$

for some constant $c \in \mathbb{R}$.

So far we have only used the information contained in Green's first theorem when we put $\phi = \psi$ for some harmonic function ψ. Instead, we now take Green's first theorem and set $\phi = 1$ on the whole of the volume $\mathcal{V}$. We then have $\boldsymbol{\nabla}\phi = 0$ and for any harmonic function ψ we find that $\int_{\mathcal{V}} \left(\phi\boldsymbol{\nabla}^2\psi + \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\psi\right) dV = 0$ and therefore $\oint_{\partial V} \boldsymbol{\nabla}\psi.d\mathbf{S} = 0$. This does of course not mean that $\boldsymbol{\nabla}\psi$ is perpendicular to $d\mathbf{S}$ everywhere but it means that the scalar product averaged along the whole boundary is 0.

74 Harmonic average
A harmonic function ψ is harmonic on the volume $\mathcal{V}$ then it satisfies

$$\oint_{\partial\mathcal{V}} \boldsymbol{\nabla}\psi.d\mathbf{S} = 0 . \tag{194}$$

[c] *Carl* Gottfried Neumann, 1832-1925.

Using [74] we can now prove an important mean value theorem for surface integrals of harmonic functions over spheres (see example 6.4 for a proof with $\mathbf{c} = \mathbf{0}$).

75 Mean value theorem for harmonic functions
A harmonic function ψ on $\mathcal{V} \subset \mathbb{R}^3$ satisfies

$$\psi(\mathbf{c}) \quad = \quad \frac{1}{4\pi R^2} \int_{\mathcal{S}_R(\mathbf{c})} \psi(\mathbf{x}) dS \ , \tag{195}$$

where $\mathcal{S}_R(\mathbf{c})$ is the surface of a sphere with radius R centred at the point $\mathbf{c}$.

Theorem [75] essentially says that the average value of a harmonic function averaged over the surface of a sphere is the same as the function value at the centre of the sphere. This immediately shows that the global maximum and the global minimum of a harmonic function defined on the volume of a sphere has to be on the boundary of the sphere. For any connected volume $\mathcal{V}$ we can now easily argue that we can fill up the volume using spheres and therefore argue that the global maximum and the global minimum of the harmonic function will have to be on the boundary $\partial\mathcal{V}$ of the volume $\mathcal{V}$.

76 Global extrema of harmonic functions
The global maximum and the global minimum of a harmonic function ψ defined on the volume $\mathcal{V}$ lies on the boundary $\partial\mathcal{V}$ unless ψ is constant on the whole of $\mathcal{V}$.

To conclude this section we now consider a harmonic function ψ defined on a volume $\mathcal{V}$ with Dirichlet boundary conditions on $\partial\mathcal{V}$: $\psi(\mathbf{x}) = f(\mathbf{x})\ \forall \mathbf{x} \in \partial\mathcal{V}$ for some given function $f(\mathbf{x})$. Let $\mathcal{F}_f$ be the set of all (differentiable) functions, not necessarily harmonic, on $\mathcal{V}$ satisfying the same Dirichlet boundary conditions on $\partial\mathcal{V}$:

$$\mathcal{F}_f \quad := \quad \{\omega : \omega(\mathbf{x}) = f(\mathbf{x})\ \ \forall \mathbf{x} \in \partial\mathcal{V}\} \ . \tag{196}$$

Obviously $\psi \in \mathcal{F}_f$ but because of [70] ψ is the only harmonic function contained in $\mathcal{F}_f$. Let us consider $\int_{\mathcal{V}} \|\boldsymbol{\nabla}\omega\|^2 dV$ for some $\omega \in \mathcal{F}_f$:

$$\begin{aligned}
\int_{\mathcal{V}} \boldsymbol{\nabla}\omega.\boldsymbol{\nabla}\omega dV &= \int_{\mathcal{V}} \boldsymbol{\nabla}(\omega - \psi + \psi).\boldsymbol{\nabla}(\omega - \psi + \psi) dV \\
&= \int_{\mathcal{V}} \boldsymbol{\nabla}(\omega - \psi).\boldsymbol{\nabla}(\omega - \psi) dV + 2\int_{\mathcal{V}} \boldsymbol{\nabla}(\omega - \psi).\boldsymbol{\nabla}\psi dV + \int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV \\
&\geq 2\int_{\mathcal{V}} \boldsymbol{\nabla}(\omega - \psi).\boldsymbol{\nabla}\psi dV + \int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV \\
&= 2\oint_{\partial\mathcal{V}} (\omega - \psi)\frac{\partial\psi}{\partial\mathbf{n}} dS - 2\int_{\mathcal{V}} (\omega - \psi)\boldsymbol{\nabla}^2\psi dV + \int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV \ ,
\end{aligned} \tag{197}$$

using Green's first theorem for the last equation. But since $\omega = \psi$ on the boundary $\partial\mathcal{V}$ and $\boldsymbol{\nabla}^2\psi = 0$ on the whole of $\mathcal{V}$ we find that the first two integrals of the last expression in EQ. 197 vanish.

77 Harmonic functions with the same Dirichlet boundary conditions
Among all (differentiable) functions $\mathcal{F}_f$ on $\mathcal{V}$ with the same Dirichlet boundary conditions $f(\mathbf{x})$ on $\partial\mathcal{V}$ as the harmonic function ψ we find that ψ minimises the integral over the norm of the gradient:

$$\int_{\mathcal{V}} \|\boldsymbol{\nabla}\omega\|^2 dV \geq \int_{\mathcal{V}} \|\boldsymbol{\nabla}\psi\|^2 dV \ , \tag{198}$$

for all functions $\omega \in \mathcal{F}_f$.

22 Gauss' flux theorem and Gauss' law

We will now return to the study of the inhomogeneous problem, namely the study of Poisson's equation $\nabla^2\phi(\mathbf{x}) = f(\mathbf{x})$ on some domain G. If we assume that ϕ is such a solution then the vector field $\mathbf{F} := \nabla\phi$ satisfies

$$\nabla.\mathbf{F} = f(\mathbf{x}) \tag{199}$$

on any volume $\mathcal{V}$ contained in G. Integrating over Eq. 199

$$\int_{\mathcal{V}} \nabla.\mathbf{F}dV = \int_{\mathcal{V}} f(\mathbf{x})dV \tag{200}$$

and applying the divergence theorem to the first of these integrals leads to *Gauss' flux theorem*[d].

78 **Gauss' flux theorem**
The function ϕ is a solution to Poisson's equation $\nabla^2\phi(\mathbf{x}) = f(\mathbf{x})$ on the domain G if and only if the vector field $\mathbf{F} := \nabla\phi$ satisfies

$$\int_{\partial\mathcal{V}} \mathbf{F}.d\mathbf{S} = \int_{\mathcal{V}} f(\mathbf{x})dV , \tag{201}$$

for any (piecewise smooth) volume $\mathcal{V} \subset G$.

We can interpret $f(\mathbf{x})$ as the sinks and sources of the vector field $\mathbf{F}$ since it corresponds to the divergence of $F(\mathbf{x})$. In other words $f(\mathbf{x})$ describes the density at which points the field starts and ends. For example, for the electric field $f(\mathbf{x})$ describes the charge density and for the gravitational field it describes the mass density. Gauss' flux theorem essentially says that *summing over the charge density on the whole volume equals the flux of the vector field through the boundary.*

We already know that $\frac{1}{r}$ is harmonic on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. If we define the corresponding vector field $\mathbf{F} = \nabla\frac{1}{r} = -\frac{\mathbf{x}}{r^3}$ then we can apply Gauss' flux theorem on some volume $\mathcal{V}$ which does not contain the origin. We then obtain $\int_{\partial\mathcal{V}} \mathbf{F}.d\mathbf{S} = 0$. But if $\mathbf{0}$ is inside the volume $\mathcal{V}$ then we take a small sphere around $\mathbf{0}$ with radius ϵ which we call $\mathcal{V}_\epsilon$ and define $\hat{\mathcal{V}}$ such that $\mathcal{V} = \hat{\mathcal{V}} \cup \mathcal{V}_\epsilon$ and $\hat{\mathcal{V}}$, $\mathcal{V}_\epsilon$ disjoint with the exception of boundary points. We then find

$$\begin{aligned}\oint_{\partial\mathcal{V}} \mathbf{F}.d\mathbf{S} &= \oint_{\partial\hat{\mathcal{V}}} \mathbf{F}.d\mathbf{S} + \oint_{\partial\mathcal{V}_\epsilon} \mathbf{F}.d\mathbf{S} \\ &= 0 + \oint_{\partial\mathcal{V}_\epsilon} \mathbf{F}.d\mathbf{S} \\ &= -\int_0^{2\pi}\int_0^{\pi} \frac{\mathbf{x}}{\epsilon^3}.\epsilon^2\mathbf{e}_r \sin\theta\, d\theta d\phi = -4\pi .\end{aligned} \tag{202}$$

The same will obviously hold if we replace $\mathbf{0}$ by any other point $\mathbf{x}_0$. This result is known as *Gauss' law.*

79 **Gauss' law**
For any volume $\mathcal{V}$ and any $\mathbf{x}_0 \in \mathbb{R}^3$ with $\mathbf{x}_0 \notin \partial\mathcal{V}$

$$\oint_{\partial\mathcal{V}} \frac{\mathbf{x} - \mathbf{x}_0}{\|\mathbf{x} - \mathbf{x}_0\|^3}.d\mathbf{S} = \begin{cases} 4\pi & , \mathbf{x}_0 \in \mathcal{V} \\ 0 & , \mathbf{x}_0 \notin \mathcal{V} \end{cases} . \tag{203}$$

[d] Conversely, if the identity of the integrals Eq. 200 holds for all possible volumes $\mathcal{V} \subset G$ then obviously the (continuous) integrands have to be the same and therefore Eq. 199 holds on G.

Gauss' law makes it clear that $\frac{\mathbf{x}-\mathbf{x}_0}{\|\mathbf{x}-\mathbf{x}_0\|^3}$ is the field of a point charge of strength 4π at the point $\mathbf{x}_0$.

23 Poisson's equation

Using the results of the previous section we are now in a position to explore the solutions to Poisson's equation. We have found that the potential $\phi = -\frac{1}{4\pi\|\mathbf{x}-\mathbf{x}_0\|}$ for the field $\mathbf{F} = \frac{1}{4\pi}\frac{\mathbf{x}-\mathbf{x}_0}{\|\mathbf{x}-\mathbf{x}_0\|^3}$ corresponds to a point charge of strength 1 at the point $\mathbf{x}$. Using the superposition principle for linear differential equations, if we had a charge of strength α at the point $\mathbf{a}$ and a charge of strength β at the point $\mathbf{b}$ the resulting solution would be just the sum

$$\mathbf{F}(\mathbf{x}) = \frac{\alpha}{4\pi}\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|^3} + \frac{\beta}{4\pi}\frac{\mathbf{x}-\mathbf{b}}{\|\mathbf{x}-\mathbf{b}\|^3}\ , \tag{204}$$

with the corresponding potential

$$\phi(\mathbf{x}) = -\frac{\alpha}{4\pi\|\mathbf{x}-\mathbf{a}\|} - \frac{\beta}{4\pi\|\mathbf{x}-\mathbf{b}\|}\ . \tag{205}$$

Adding up finitely many charges of strengths α_i at the positions $\mathbf{a}_i$ we would obviously find

$$\phi(\mathbf{x}) = -\sum_i \frac{\alpha_i}{4\pi\|\mathbf{x}-\mathbf{a}_i\|}\ . \tag{206}$$

In case the charges are not located at finitely many points but are described by a charge density we expect that a suitable limit process will result in the sum in Eq. 206 to be replaced by an integral and the charge strengths α_i will go over to the charge density $f(\mathbf{x})$. We will give this result without proof but it can in fact easily be shown by substituting Eq. 208 into Gauss' flux theorem Eq. 201, exchanging the order of integration and using Gauss' law. We then obtain a solution to Poisson's equation with $\phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$ (see the third method in example 6.5).

80 Solutions to Poisson's equation
For Poisson's equation $\nabla^2\phi = f(\mathbf{x})$ defined on some finite volume $\mathcal{V}$ we can set $f(\mathbf{x}) = 0$ outside the volume $\mathcal{V}$ and obtain a solution[e]

$$\phi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\, d(y_1, y_2, y_3)\ , \tag{207}$$

where the integral is taken over the whole $\mathbb{R}^3$. This solution satisfies the boundary conditions $\phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. The corresponding vector field is given by

$$\mathbf{F}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\mathbf{y})(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^3}\, d(y_1, y_2, y_3)\ . \tag{208}$$

[80] gives of course only one particular solution with the boundary conditions $\phi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. If a specific problem needs to be solved under different boundary conditions then solutions of Laplace's equation have to be added (as solutions of the homogeneous problem) in order to achieve the required boundary conditions.

[e] Note that $\int_{\mathbb{R}^3}$ is in fact only an integral $\int_{\mathcal{V}}$.

Examples

Example 6.1 *Maxwell's equations for electromagnetic fields are*

$$\nabla . \mathbf{E} = \frac{\rho}{\epsilon_o} \tag{209}$$

$$\nabla . \mathbf{B} = 0 \tag{210}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{211}$$

$$\nabla \times \mathbf{B} = \epsilon_o \mu_o \frac{\partial \mathbf{E}}{\partial t} + \mu_o \mathbf{j} \tag{212}$$

Here $\mathbf{E}$ *and* $\mathbf{B}$ *are the electric and magnetic fields;* ρ *and* $\mathbf{j}$ *are electric charge density and current density;* ϵ_o *and* μ_o *are constants (the permitivity and permissivity of free space) and they satisfy* $\epsilon_o \mu_o = c^{-2}$, *where* c *is the speed of light. (This last equation shows that there is a deep connection between electromagnetic fields and light.) All fields may depend on both* $\mathbf{x}$ *and time* t.

Note that there is some symmetry between the equations. Equations (1) and (2) are similar except that (2) has no source on the right hand side. This is as expected, because the magnetic charge density is zero (there are no magnetic monopoles, at least in classical theory). Similarly, equations (3) and (4) are the similar, expect for the absence of magnetic current in (3), and an extra minus sign.

It is worth pausing for a moment to investigate the integral forms of these equations; it is a nice example of the use of the divergence theorem and Stokes' theorem and relates the equations to physics that you might know, but it is not essential to the example.

If we integrate (1) over a volume V *with surface* S, *we get*

$$\int_V \nabla . \mathbf{E}\, dV = \frac{1}{\epsilon_o} \int_V \rho(\mathbf{x}) dV \qquad i.e. \qquad \int_S \mathbf{E} . d\mathbf{S} = Q/\epsilon_o$$

or, in words, the flux of electric field across any closed surface is equal to the total charge within the surface over ϵ_o. *This is Gauss's law. The same calculation for equation (2) shows that the total flux of magnetic field across a closed surface is zero.*

If we integrate (3) over an open surface S *with boundary curve* C, *we get*

$$\int_S \nabla \times \mathbf{E} . d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S} \qquad i.e. \qquad \int_C \mathbf{E} . d\mathbf{x} = -\frac{d}{dt} \int_S \mathbf{B} . d\mathbf{S}.$$

In words, this says that the circulation of the electric field (which is called the the EMF or electromotive force) round a close curve is equal to the rate of change of flux through the curve. This Faraday's law of induction (and the minus sign relates to Lenz's Law, about the effect tending to oppose the cause).

Finally, if we integrate (4) over an open surface S *with boundary curve* C, *ignoring the first term on the right hand side (which would be a valid approximation for situations in which* c^{-2} *is comparatively small), we get*

$$\int_S \nabla \times \mathbf{B} . d\mathbf{S} = \mu_o \int_S \mathbf{j} . d\mathbf{S} \qquad i.e. \qquad \int_C \mathbf{B} . d\mathbf{x} = \mu_o I.$$

In words, this says that the circulation of magnetic field round a loop is equal to the μ_o *times the current through the loop. This is Ampère's law.*

Going back to the original equations (1)—(4), we see that the structure of these equations allows potentials to be constructed. The easiest case is when the fields are time independent (all time derivatives are zero). In this case, equation (3) becomes $\nabla \times \mathbf{E} = \mathbf{0}$, *which implies that there exists a scalar potential* ϕ *such that*

$$\mathbf{E} = -\nabla\phi$$

(the minus sign is conventional) and equation (1) then shows that ϕ *satisfies Poisson's equation*

$$\nabla . (-\nabla\phi) = -\nabla^2\phi = \frac{\rho}{\epsilon_o}$$

This is useful, because a great deal is known about solutions of Poisson's equation.

If the fields are time-dependent, there is no scalar potential for $\mathbf{E}$, *since* $\nabla \times \mathbf{E} \neq \mathbf{0}$. *However, equation (2) shows that we can always find a vector potential for* $\mathbf{B}$*:*

$$\nabla . \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}.$$

Of course, $\mathbf{A}$ *is not uniquely determined: we can still add the gradient of any scalar field without changing* $\mathbf{B}$. *We can use this freedom is to choose* $\mathbf{A}$ *to satisfy* $\nabla . \mathbf{A} = 0$. *(If* $\nabla . \mathbf{A} \neq 0$, *we need to add* $\nabla\chi$ *where* χ *satisfies* $\nabla . (\mathbf{A} + \nabla\chi) = 0$, *i.e.* $\nabla^2\chi = -\nabla . \mathbf{A}$. *This is Poisson's equation, and, for any given* $\nabla . \mathbf{A}$, *there is always a solution for* χ.*)*

Thus equation (3) becomes

$$\nabla \times \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t} \qquad i.e. \qquad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

This means that we can find a scalar potential for $\mathbf{E} + \dfrac{\partial \mathbf{A}}{\partial t}$ *:*

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi \qquad or \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$$

Why are these potentials useful? If we write $\mathbf{E}$ *and* $\mathbf{B}$ *in terms of potentials, then Maxwell equations (2) and (3) are automatically satisfied. The remaining equations become*

$$\nabla . \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) = \frac{\rho}{\epsilon_o} \qquad i.e. \qquad \nabla^2\phi = -\frac{\rho}{\epsilon_o}$$

(using the gauge condition $\nabla . \mathbf{A} = 0$*), and*

$$\nabla \times (\nabla \times \mathbf{A}) = \epsilon_o\mu_o\frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) + \mu_o\mathbf{j}.$$

In the absence of sources, i.e. $\rho = 0$ *and* $\mathbf{j} = 0$, *the potential* $\phi = 0$ *and this last equation reduces to*

$$\nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}.$$

This is the wave equation; it says that electromagnetic fields propagate like waves with speed c — i.e. with the speed of light.

Example 6.2 *We will prove a uniqueness theorem for solutions for the differential equation*

$$y'' - y = f(x),$$

with Dirichlet boundary conditions

$$y(0) = a; \quad y(1) = b.$$

The method used is the basis for all other uniqueness theorems.

Suppose that the above boundary value problem has two solutions, $y_1(x)$ and $y_2(x)$, and let $Y(x) = y_2(x) - y_1(x)$. Then $Y(x)$ satisfies the equation $Y'' - Y = 0$ and the boundary conditions $Y(0) = Y(1) = 0$. We will show that $Y(x) \equiv 0$.

The technique is to show that, if $Y(x) \neq 0$, then a positive quantity is negative. The positive quantity, I, is given by

$$I = \int_0^1 (Y')^2 dx.$$

Note that

$$I = 0 \Leftrightarrow Y'(x) = 0 \Leftrightarrow Y(x) = \text{ constant} \Leftrightarrow Y(x) = 0.$$

The last implication follows because $Y(0) = 0$, so if $Y(x)$ is constant, its value must be zero.

To obtain the contradiction, we integrate by parts:

$$I = \int_0^1 Y'(x)Y'(x)dx = Y(x)Y'(x)\Big|_0^1 - \int_0^1 Y(x)Y''(x)dx = -\int_0^1 Y(x)^2 dx \leq 0\,.$$

For the last equality we have used the boundary conditions and the differential equation for Y. Thus $I \leq 0$, which is the required contradiction.

Example 6.3 *We will find separable solutions of Laplace's equation of the form $f(x, y) = X(x)Y(y)$.*

Substituting into Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

gives

$$X''(x)Y(y) + X(x)Y''(y) = 0\,,$$

or, after rearrangement,

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x} = \underbrace{-\frac{Y''(y)}{Y(y)}}_{\text{function of } y} = C\,,$$

where C is a constant. This is the critical argument: the only function of x that equals a function of y is a constant. The constant C is not undetermined so far, but will be determined by boundary conditions.

There are three cases to consider.

(i): $C = 0$.

In this case

$$X''(x) = Y''(y) = 0 \quad \Rightarrow \quad X(x) = ax + b \quad \text{and} \quad Y(y) = cy + d\,,$$

where a, b, c and d are constants, i.e.

$$f(x, y) = (ax + b)(cy + d)\,.$$

(ii) *: $C = k^2 > 0$.*

In this case

$$X'' - k^2 X = 0\,, \quad \textit{and} \quad Y'' + k^2 Y = 0\,.$$

Hence

$$X(x) = a e^{kx} + b e^{-kx}\,, \quad \textit{and} \quad Y(y) = c \sin ky + d \cos ky\,,$$

where a, b, c and d are constants, i.e.

$$f(x, y) = (a e^{kx} + b e^{-kx})(c \sin ky + d \cos ky)\,.$$

(iii) *$C = -k^2 < 0$.*

This is similar to (ii) above, with $x \leftrightarrow y$:

$$f(x, y) = (a e^{ky} + b e^{-ky})(c \sin kx + d \cos kx)\,.$$

The question of which of the above solutions (or which linear combination of them) is appropriate is decided by the boundary conditions. Obviously, if the solution is periodic in x, or has to vanish at two values of x, we must have $C < 0$ so that the solutions are of the form (iii).

Example 6.4 *We will prove a Mean Value Theorem for harmonic functions. (Note that a harmonic function is one that satisfies Laplace's equation.)*

Let the scalar field φ be harmonic in a volume $\mathcal{V}$ bounded by a closed surface $\mathcal{S}$. Consider the function $f(r)$ defined to be the mean value of φ on a spherical surface $\mathcal{S}_r$ given by $\|\mathbf{x}\| = r$, i.e.

$$\begin{aligned} f(r) &= \frac{1}{4\pi r^2} \int_{\mathcal{S}_r} \varphi(\mathbf{x})\, \mathrm{d}S \\ &= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \varphi(r, \theta, \phi) \sin\theta \, d\phi \, d\theta\,, \qquad (*) \end{aligned}$$

where (r, θ, ϕ) are spherical polar co-ordinates. We will show that f is a constant, and that $f(r) = \varphi(\mathbf{0})$.

We can show this result either by using Green's Second Theorem, or directly as follows:

$$\begin{aligned} \frac{df}{dr} &= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\partial \varphi}{\partial r} \sin\theta \, d\phi \, d\theta \\ &= \frac{1}{4\pi r^2} \int_{\mathcal{S}_r} \frac{\partial \varphi}{\partial r}\, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \int_{\mathcal{S}_r} \boldsymbol{\nabla}\varphi . \widehat{\mathbf{x}}\, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \int_{\mathcal{S}_r} \boldsymbol{\nabla}\varphi . \mathrm{d}\mathbf{S} \\ &= \frac{1}{4\pi r^2} \int_{\mathcal{V}} \nabla^2 \varphi \, \mathrm{d}V \\ &= 0\,. \end{aligned}$$

Further, setting $r = 0$ in $(*)$ shows that

$$f(0) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \varphi(0, \theta, \phi) \sin\theta \, d\phi \, d\theta = \varphi(\mathbf{0})\,,$$

and hence, since $f(r)$ is constant,

$$\varphi(\mathbf{0}) = f(0) = f(r) = \frac{1}{4\pi r^2}\int_{\mathcal{S}_r} \varphi(\mathbf{x})\,\mathrm{d}S\,.$$

We conclude that:

> *if φ is harmonic, the value of φ at a point is equal to the average of the values of φ on any spherical shell centred at that point.*

Example 6.5 *We will obtain the gravitational field due to a sphere $\|\mathbf{x}\| = R$ of uniform density ρ_0, and total mass $M = \frac{4}{3}\pi R^3\rho_0$ in two ways.*

First way: solve Poisson's Equation.

We have

$$\nabla^2\varphi = \begin{cases} 4\pi G\rho_0 & \textit{for} \quad r < R \\ 0 & \textit{for} \quad r > R. \end{cases}$$

We assume (or could prove) that φ and $\frac{\partial\varphi}{\partial n}$ are continuous at $r = R$. From symmetry considerations we anticipate (or we could prove, e.g. by seeking separable solutions) that φ will be a function only of r, i.e. $\varphi \equiv \varphi(r)$.

Using the expression for the Laplacian in spherical polar co-ordinates, for $r < R$ we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 4\pi G\rho_0\,,$$

which we can integrate:

$$r^2\frac{d\varphi}{dr} = \tfrac{4}{3}\pi G\rho_0 r^3 + a\,,$$

and

$$\varphi = \tfrac{2}{3}\pi G\rho_0 r^2 - \frac{a}{r} + b\,,$$

where a and b are constants. Since there is no point mass at the origin, φ is bounded there, and hence $a = 0$.

Similarly, for $r > R$ we deduce that

$$\varphi = -\frac{d}{r} + c\,,$$

where c and d are constants. Since the potential φ is only defined up to a constant, we can set $c = 0$; this means that $\varphi(r) \to 0$ as $r \to \infty$.

Now we match the interior and exterior solutions at $r = R$ to find b and d:

$$\tfrac{2}{3}\pi G\rho_0 R^2 + b = -\frac{d}{R} \qquad \textit{and} \qquad \tfrac{4}{3}\pi G\rho_0 R = \frac{d}{R^2}\,,$$

so

$$d = \tfrac{4}{3}\pi G\rho_0 R^3 \qquad \textit{and} \qquad b = -2\pi G\rho_0 R^2\,.$$

Thus

$$\varphi = -\frac{4\pi G\rho_0 R^3}{3r}\,, \qquad \mathbf{g} = -\boldsymbol{\nabla}\varphi = -\frac{4\pi G\rho_0 R^3}{3r^2}\,\widehat{\mathbf{x}} = -\frac{GM}{r^2}\,\widehat{\mathbf{x}} \quad \textit{for} \quad r \geq R\,,$$

$$\varphi = \tfrac{2}{3}\pi G\rho_0(r^2 - 3R^2)\,, \qquad \mathbf{g} = -\boldsymbol{\nabla}\varphi = -\tfrac{4}{3}\pi G\rho_0 r\,\widehat{\mathbf{x}} = -\frac{GMr}{R^3}\,\widehat{\mathbf{x}} \quad \textit{for} \quad r \leq R\,.$$

Note that outside the sphere the gravitational acceleration is equal to that of a particle of mass M. Within the sphere, the gravitational acceleration increases linearly with distance from the centre.

Second way: use Gauss' Flux Theorem.

From symmetry consideration we anticipate that $\mathbf{g}$ *will be a function only of* r*, and will be parallel to* $\widehat{\mathbf{x}}$*, i.e.* $\mathbf{g} = g(r)\widehat{\mathbf{x}}$*. Suppose* $\mathcal{S}$ *is a spherical surface of radius* r*. Then*

$$\int_{\mathcal{S}} \mathbf{g}.\,d\mathbf{S} = \int_{\mathcal{S}} g(r)\,\widehat{\mathbf{x}}.\,d\mathbf{S} = g(r)\,4\pi r^2\,.$$

Further,

$$\begin{array}{llll} \textit{if} \quad r \geq R & \textit{then} & \displaystyle\int_{\mathcal{V}} \rho\,dV = \tfrac{4}{3}\pi R^3 \rho_0 = M\,, \\ \textit{if} \quad r \leq R & \textit{then} & \displaystyle\int_{\mathcal{V}} \rho\,dV = \tfrac{4}{3}\pi r^3 \rho_0 = Mr^3/R^3\,. \end{array}$$

Hence from Gauss' flux theorem

$$g(r) = \begin{cases} -GM/r^2 & r \geq a \\ -GMr/R^3 & r \leq a \end{cases}.$$

We can obtain $\varphi(r)$ *by integrating* $g(r) = -d\varphi/dr$*, using the boundary condition* $\varphi = 0$ *at* $r = \infty$ *and continuity of* $\varphi(r)$ *at* $r = R$*.*

Third way: use the general solution of Poisson's equation.

The potential at a fixed point $\mathbf{X}$ *due to a mass distribution* $\rho(\mathbf{x})$ *in volume* V *is given by*

$$\varphi(\mathbf{X}) = G\int_{\mathcal{V}} \frac{\rho(\mathbf{x})}{\|\mathbf{X} - \mathbf{x}\|}\,dV\,.$$

We choose polar coordinates with pole direction $\mathbf{X}$*, so that* $\|\mathbf{X} - \mathbf{x}\| = (d^2 + r^2 - 2rd\cos\theta)^{\frac{1}{2}}$*, where* $d = \|\mathbf{X}\|$ *and* (r, θ, ϕ) *are the polar coordinates of the point* $\mathbf{x}$*. Then*

$$\begin{aligned} \varphi(\mathbf{X}) &= G\int_0^R\int_0^\pi\int_0^{2\pi} \frac{\rho_0}{(d^2 + r^2 - 2rd\cos\theta)^{\frac{1}{2}}}\,r^2\sin\theta d\phi d\theta dr \\ &= 2\pi G\rho_0 \int_0^R\int_0^\pi \frac{r^2\sin\theta}{(d^2 + r^2 - 2rd\cos\theta)^{\frac{1}{2}}} d\theta dr \\ &= 2\pi G\rho_0 \int_0^R \frac{r}{d}\left(|d + r| - |d - r|\right) dr\,. \end{aligned}$$

If $d > R$ *(i.e. if* $\mathbf{X}$ *is outside the massive sphere), then the integrand is* $2r^2/d$*, which gives* $\varphi(d) = GM/d$*. If* $d < R$*, then the integrand is* $2r^2/d$ *if* $r < d$ *but* $2r$ *if* $r > d$*. In this case,*

$$\varphi(d) = 2\pi G\rho_0 \int_0^d \frac{2r^2}{d} dr + 2\pi G\rho_0 \int_d^R 2r dr$$

which gives the required result.

Remark *Similar results hold for the electric field due to a uniformly charged sphere.*

This is because the electric field $\mathbf{E}$ *due to a point charge* q_1 *at* $\mathbf{x}_1$ *is also given by an inverse square law,*

$$\mathbf{E} = \frac{q_1}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_1}{\|\mathbf{x} - \mathbf{x}_1\|^3},$$

where ε_0 *is the permittivity of free space. Thus we can read off the equivalent results for electrostatics by means of the transformations*

$$\mathbf{g} \to \mathbf{E}, \quad m_j \to q_j, \quad G \to -\frac{1}{4\pi\varepsilon_0}.$$

For instance, Gauss' flux theorem becomes

$$\int_{\mathcal{S}} \mathbf{E}.\, d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\mathcal{V}} \rho \; dV,$$

where ρ *is the now the charge density. The differential form of this equation is*

$$\boldsymbol{\nabla}.\, \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

which is one of Maxwell's equations. Further, if we introduce the electric potential φ*, where*

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi,$$

then

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon_0},$$

with solution in unbounded space

$$\varphi(\mathbf{X}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{x})}{\|\mathbf{X} - \mathbf{x}\|} \, dV.$$